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An Additive Theory of the Zeros of the Riemann Zeta Function

by

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§1. Introduction

The purpose of the present article is to study the distribution of the sums of the zeros of the Riemann zeta function $\zeta(s)$. The results were first announced in [13]. In the meantime, some of the results were refined and announced in [14] and further new and refined results have been obtained since then. Here we shall give the proofs to these with the full details and in the refined forms.

While much studies on the distribution of the zeros of $\zeta(s)$ are well-known and its control of the distribution of the prime numbers have attracted many mathematicians, there seems to be quite few study on the distribution of the sums of the zeros of $\zeta(s)$. So it might be not worthless to describe our motivations, at this moment, to study this subject.

(I). First of all, we want to see how the control by the distribution of the zeros of $\zeta(s)$ of the distribution of the prime numbers will be changed or unchanged under the addition of the zeros.

More concretely, as the relations between the zeros and the primes, we have in mind the following three types of explicit formulas.

(a): Riemann's explicit formula, namely,

$$\sum_{n\le X}{}' \Lambda(n)=X-\sum_{\rho}\frac{X^{\rho}}{\rho}-\frac{\zeta'}{\zeta}(0)-\frac{1}{2}\log(1-X^{-2}),$$

where $X>1$, ρ runs over all the non-trivial zeros of $\zeta(s)$, the dash on the sum indicates that we count with the multiplicity 1/2 for $n=X$ and we put

$$\Lambda(x)=\begin{cases}\log p & \text{if } x=p^k \text{ with a prime number } p \text{ and an integer } k\ge 1\\ 0 & \text{otherwise}.\end{cases}$$

(b): Landau's explicit formula, which will be described below as **(B)**.

(c): (On R.H.)

$$\sum_{1\le n\le X}\Lambda(n)e(-\alpha n)=-e^{(-\pi/4)i}\frac{1}{\sqrt{\alpha}}\sum_{0<\gamma\le 2\pi\alpha X}e^{i\gamma\log(\gamma/2\pi e\alpha)}+O\left(\sqrt{X}\left(\frac{\log X}{\log\log X}\right)^{2}\right),$$

where $X\gg 1$, $e(t)$ denotes $e^{2\pi it}$, α is a positive number, γ runs over the imaginary parts of the zeros of $\zeta(s)$ and R.H. is the abbreviation the Riemann Hypothesis.

(c) is a consequence of **(C″)** with $b=1$ below. Thus we are concerned with the problem to replace ρ by $\rho+\rho'$ or γ by $\gamma+\gamma'$ in the above **(a)**, **(b)** and **(c)**, where ρ' runs also over the non-trivial zeros of $\zeta(s)$ and γ' runs over the imaginary parts of the zeros of $\zeta(s)$. We could say, as will be seen below, that we are successful in the case of **(b)** or **(c)**, however we shall leave the extension of **(a)** as an open problem.

(II). Second, one has conjectured that the positive imaginary parts of the zeros of $\zeta(s)$ are linearly independent over the rationals (or, even strongly, algebraically independent). This implies, in particular, that

the number of the ways of the expressions of a number by the sums of the imaginary parts $\gamma+\gamma'$ is essentially one.

It seems to be interesting even to prove the last statement.

To understand the strength of such assumption as the linear independence of the positive imaginary parts of the zeros of $\zeta(s)$, we may recall two consequences.

Ingham [24] showed that

$$\overline{\lim_{X\to\infty}}\frac{M(X)}{\sqrt{X}}=+\infty \quad\text{and}\quad \underline{\lim_{X\to\infty}}\frac{M(X)}{\sqrt{X}}=-\infty\,,$$

where

$$M(X)=\sum_{n\le X}\mu(n)$$

with the Möbius function $\mu(n)$.

On the other hand, the author [17] has determined explicitly the value distribution of the quantity

$$\sum_{\gamma>0}\frac{X^{i\gamma}}{(1/2+i\gamma)(3/2+i\gamma)}$$

which appears as the oscillating term in the mean value theorem on Goldbach's problem

$$\sum_{n\le X}\left\{\sum_{m+k=n}\Lambda(m)\Lambda(k)-n\cdot\prod_{p|n}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid n}\left(1-\frac{1}{(p-1)^2}\right)\right\}$$
$$=-4X^{3/2}\Re\left\{\sum_{\gamma>0}\frac{X^{i\gamma}}{(1/2+i\gamma)(3/2+i\gamma)}\right\}+O((X\log X)^{1+1/3})$$

or the mean value theorem on the prime number theorem

$$\int_0^X\left(\sum_{n\le y}\Lambda(n)-y\right)dy$$
$$=-2X^{3/2}\Re\left\{\sum_{\gamma>0}\frac{X^{i\gamma}}{(1/2+i\gamma)(3/2+i\gamma)}\right\}-X\log(2\pi)+\log(2\pi)+C_o$$

$$-1-\frac{6}{\pi^2}\zeta'(2)-X\sum_{n=1}^{\infty}\frac{X^{-2n}}{2n(2n-1)},$$

for example, where C_o is the Euler constant.

It is well-known that the Riemann Hypothesis is equivalent to the statement that

$$M(X)=O(X^{1/2+\varepsilon}) \qquad \text{for any} \quad \varepsilon>0\,.$$

And that the Mertens' Conjecture

$$|M(X)|<X^{1/2} \qquad \text{for any} \quad X>1$$

was disproved by Odlyzko and te Riele [29].

(III). Finally, the recent works by Kurokawa [25] and Deninger [2] seems to try to study a function whose zeros are exactly the sums of the zeros of $\zeta(s)$. If it is successful, it should fix our problem left opent just above in **(I)**.

In this article, we shall give a study mainly from the first point of view.

We start with recalling some of the known results concerning the distribution of the zeros of $\zeta(s)$ in order to have a target to attack at hand. We recall first the well known Riemann-von Mangoldt formula for the number $N(T)$, where $N(T)$ denotes the number of the zeros $\beta+i\gamma$ of $\zeta(s)$ in $0<\gamma\leq T$ and $0<\beta<1$. Let

$$S(T)=\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+iT\right) \qquad \text{for} \quad T\neq\gamma\,,$$

where the argument is obtained by the continuous variation along the straight lines joining 2, $2+iT$, and $1/2+iT$, starting with the value zero. When $T=\gamma$, then we put

$$S(T)=S(T+0)\,.$$

Then the well known Riemann-von Mangoldt formula (cf. p. 212 of Titchmarsh [35]) states that

$$\textbf{(A)}: \qquad N(T)=\frac{1}{\pi}\vartheta(T)+1+S(T)\,,$$

where $\vartheta(T)$ is the continuous function defined by

$$\vartheta(T)=\Im\left(\log\Gamma\left(\frac{1}{4}+\frac{iT}{2}\right)\right)-\frac{1}{2}T\log\pi$$

with

$$\vartheta(0)=0\,,$$

$\Gamma(s)$ being the Γ-function.

We know that

$$\vartheta(T)=\frac{T}{2}\log\frac{T}{2\pi}-\frac{T}{2}-\frac{\pi}{8}+\frac{1}{48T}+\frac{7}{5760T^3}+\cdots$$

and that

$$S(T)\ll\log T.$$

It is a well known theorem of Littlewood (cf. 256 of Titchmarsh [35]) that under the Riemann Hypothesis we have

$$S(T)=O\left(\frac{\log T}{\log\log T}\right).$$

It is also well-known that

$$\int_0^T S(t)dt=O(\log T)$$

and that with some constant A,

$$\int_0^T \frac{S(t)}{t}dt=A+O\left(\frac{\log T}{T}\right).$$

The latter was proved by F. and R. Nevalinna (cf. p. 222 of Titchmarsh [35]).

We recall second Landau's theorem on the arithmetic connection of the zeros with a prime number (cf. Landau [26]).

(B): $$\sum_{0<\gamma\leq T} X^{\rho}=-\frac{T}{2\pi}\Lambda(X)+O(\log T)$$

for any $X>1$ *and* $T>T_o$.

This sum picks up one prime number. It implies also, by Weyl criterion, that

γ_n *is uniformly distributed mod one* ,

where γ_n is the n-th positive imaginary part of the zeros of $\zeta(s)$.

The author [11] [14] has refined Landau's formula under R.H. as follows.

(B′): (On R.H.)

$$\sum_{0<\gamma\leq T} X^{i\gamma}=-\frac{T}{2\pi}\frac{\Lambda(X)}{\sqrt{X}}+\frac{X^{iT}\log\frac{T}{2\pi}}{2\pi i\log X}+X^{iT}S(T)+O\left(\frac{\log T}{(\log\log T)^2}\right).$$

The explicit dependence on X in **(B)** or **(B′)** is also important in some applications as we shall also see in the present article. It has been studied by Gonek [22] and Fujii [11] [14]. (cf. Lammas 1 and 2 below.)

We recall next the following result on an arithmetic connection of the zeros with a rational number (cf. Fujii [4] [5] [6]).

(**C**): (On R.H.)

For any positive α, *we have*

$$\lim_{T\to\infty} \frac{1}{\frac{T}{2\pi}} \sum_{0<\gamma\le T} e\left(\frac{\gamma}{2\pi}\log\frac{\gamma}{2\pi e\alpha}\right)$$

$$=\begin{cases} -e^{(\pi/4)i}C\left(\frac{a}{q}\right) & \text{if } \alpha=\frac{a}{q} \text{ with integers } a \text{ and } q\ge 1,\ (a,q)=1 \\ 0 & \text{if } \alpha \text{ is irrational}, \end{cases}$$

where we put

$$C\left(\frac{a}{q}\right)=\varphi(q)^{-1}\left(\frac{a}{q}\right)^{-1/2}\bar{S}\left(\frac{a}{q}\right),$$

$$S\left(\frac{a}{q}\right)=\sum_{b=1,(b,q)=1}^{q} e\left(\frac{a}{q}b\right)=\mu(q)$$

and $\varphi(q)$ *is the Euler function.*

(**C**) has been generalized in Fujii [10] as follows.

(**C′**): (On R.H.)

Let K be an integer ≥ 1. *Then for any positive* α, *we have*

$$\lim_{T\to\infty} \frac{1}{\left(\frac{T}{2\pi}\right)^{(1/2)(1+1/K)}} \sum_{0<\gamma\le T} e\left(\frac{\gamma}{2\pi K}\log\frac{\gamma}{2\pi e\alpha K}\right)$$

$$=\begin{cases} -e^{(\pi/4)i}C\left(\frac{a}{q},K\right) & \text{if } \alpha=\frac{a}{q} \text{ with integers } a \text{ and } q\ge 1,\ (a,q)=1 \\ 0 & \text{if } \alpha \text{ is irrational}, \end{cases}$$

where we put

$$C\left(\frac{a}{q},K\right)=2K^{(1/2)(1-1/K)}\frac{1}{(K+1)\varphi(q)}\left(\frac{a}{q}\right)^{-1/2K}\bar{S}\left(\frac{a}{q},K\right)$$

and

$$S\left(\frac{a}{q},K\right)=\sum_{b=1,(b,q)=1}^{q} e\left(\frac{a}{q}b^K\right).$$

Thus such singular property at rational α in (**C**) is not changed by replacing γ by γ/K, $K=2,3,4,\cdots$. However the conjecture stated in (**II**) above suggests that

$\frac{\gamma}{K}$ *will never be an imaginary part of the zeros of* $\zeta(s)$ *for* $K=2,3,4,\cdots$.

(**C**′) is a consequence of the following result (cf. Fujii [16]), under R.H., which improves upon our previous results in Fujii [6].

(**C**″): (On R.H.)

For $0<\alpha\ll T$ *and* $b>0$, *we have*

$$\sum_{0<\gamma\le T} e^{ib\gamma\log(b\gamma/2\pi e\alpha)} = -e^{(\pi/4)i}\frac{\sqrt{\alpha}}{b}\sum_{1\le n\le (Tb/2\pi\alpha)^b}\frac{\Lambda(n)}{n^{1/2-1/2b}}e(-\alpha n^{1/b})$$

$$+O\left(\left(\frac{T}{\alpha}\right)^{b/2}\log\left(\frac{T}{\alpha}\right)\frac{\log T}{(\log\log T)^2}\right)+O(T^{1/2-b/2}\alpha^{b/2})+O((\sqrt{\alpha}+1)\log T+\alpha^{b/2}).$$

This implies, using Piateski-Shapiro [30], also that

$$\sum_{0<\gamma\le T} e^{ib\gamma\log(b\gamma/2\pi e\alpha)}=o(T\log T)$$

for $0<b<6/5$. We expect certainly the following conjecture:

For any $b>0$ *and any positive* α, *we have*

$$\sum_{0<\gamma\le T} e^{ib\gamma\log(\gamma/2\pi e\alpha)}=o(T\log T).$$

This implies, by Weyl criterion, that for any positive b and positive α,

$$b\gamma_n\log\frac{\gamma_n}{2\pi e\alpha}\ \textit{is uniformly distributed mod one}.$$

As another aspect of the above sum, we recall the following theorem which shows that the vertical distribution of the zeros of $\zeta(s)$ has a deep connection with the Generalized Riemann Hypothesis (G.R.H.) for the Dirichlet L-functions $L(s,\chi)$ (cf. Fujii [10][16]).

(**D**): (On R.H.)

For any integer $K\ge 1$, *G.R.H. for all Dirichlet L-function* $L(s,\chi^K)$ *with a Dirichlet character* χ *mod* $q\ge 3$ *is equivalent to the relation*

$$\sum_{0<\gamma\le T} e\left(\frac{\gamma}{2\pi K}\log\frac{\gamma}{2\pi eK\frac{a}{q}}\right)=-e^{(\pi/4)i}C\left(\frac{a}{q},K\right)\left(\frac{T}{2\pi}\right)^{(1/2)(1+1/K)}+O(T^{1/2+\varepsilon})$$

for any positive ε *and any integer* a *satisfying* $1\le a\le q$ *and* $(a,q)=1$ *and* $T>T_0$.

We should notice that we have proved (**D**) for $K\ge 5$ in [9][10]. The result for $K=1$, 2, 3 and 4 comes from the improvement, as described in (**C**″), on the remainder term of the exponential sum. An infinit series version of the above result for $K=1$ was shown by Sprindzuk [34]. The author [9] [10] has generalized it in two directions as follows.

(D′): (On R.H.)

For any integer $K \geq 1$, *G.R.H. for all* $L(s, \chi^K)$ *with a character* χ *mod* $q \geq 3$ *is equivalent to the relation*

$$\sum_{\gamma>0} e\left(\frac{\gamma}{2\pi K}\log\frac{\gamma}{eK}\right)e^{-\gamma\pi/2K}\gamma^{(1/2)((1/K)-1)}\left(x+2\pi i\frac{a}{q}\right)^{-(1/K)(1/2+i\gamma)}$$

$$=-\hat{C}\left(\frac{a}{q}, K\right)X^{-1/K}+O(X^{-1/2K-\varepsilon})$$

as $X \to +0$ *for any positive* ε *and for any integer* a *with* $1 \leq a \leq q$ *and* $(a, q)=1$, *where we put*

$$\hat{C}\left(\frac{a}{q}, K\right)=e^{(\pi i/4)(1-1/K)}\Gamma\left(\frac{1}{K}\right)K^{(1/2)((1/K)-1)}\overline{S}\left(\frac{a}{q}, K\right)\frac{1}{\sqrt{2\pi\varphi(q)}}.$$

(D″): (On R.H.)

For any integer $K \geq 2$, *G.R.H. for all* $L(s, \chi)$ *with a character* χ *mod* $q \geq 3$ *is equivalent to the relation*

$$\sum_{\gamma>0} e\left(\frac{K\gamma}{2\pi}\log\frac{K\gamma}{e}\right)e^{-K\gamma\pi/2}\gamma^{(1/2)(K-1)}\left(x+2\pi i\frac{a}{q}\right)^{-K(1/2+i\gamma)}\left(1+\frac{A_1}{\gamma}+\frac{A_2}{\gamma^2}+\cdots+\frac{A_{K_0}}{\gamma^{K_0}}\right)$$

$$+B(K)\sum_{d=1,d\neq K}^{2K-1}\sum_p \log p\cdot e^{-xp^{d/K}}e\left(-\frac{a}{q}p^{d/K}\right)$$

$$=-\frac{1}{X}B(K)\frac{\mu(q)}{\varphi(q)}+O(X^{-1/2-\varepsilon})$$

as $X \to +0$ *for any positive* ε *and for any integer* a *with* $1 \leq a \leq q$ *and* $(a, q)=1$, *where,*

$$B(K)=e^{(\pi i/4)(1-K)}K^{-(1/2)(K+1)}\frac{1}{\sqrt{2\pi}},$$

$K_o=[(1/2)(K-1)]$ *if* $K \geq 3$, $A_1, \cdots, A_{K_o}$ *are the constants which may depend on* K, $A_1=\cdots=A_{K_o}=0$ *if* $K=2$.

(D′) for $K=1$ is the Sprindzuk's result [34]. Thus we have obtained some equivalent conditions for G.R.H. in terms of finite or infinit sums involving the zeros of $\zeta(s)$ under R.H.

We are now in a position to state our problem which we shall study in this article.

PROBLEM. *Extend* **(A)**, **(B)**, **(B′)**, **(C)**, **(C′)** *and* **(D)** *to the sums of the zeros. In particular, study how* **(A)**, **(B)**, **(B′)**, **(C)**, **(C′)** *and* **(D)** *will be changed under the addition of the zeros.*

The first task is to get the Riemann-von Mangoldt formula for the number

$$N_2(T)=\#\{0<\gamma+\gamma' \leq T;\ 0<\gamma, \gamma' \leq T\}.$$

We shall prove the following theorem.

THEOREM 1. *For $T>T_o$, we have*

$$N_2(T)=L_2(T)+R_2(T)\,,$$

where we put

$$L_2(T)=\frac{1}{8\pi^2}T^2\log^2 T-\frac{1}{8\pi^2}T^2\log T\{3+2\log(2\pi)\}$$

$$+\frac{1}{16\pi^2}T^2\{7+6\log(2\pi)+2\log^2(2\pi)-2\zeta(2)\}$$

$$+C_1 T\log T+C_2 T+O(\log^2 T)$$

with some constants C_1 and C_2 and we put

$$R_2(T)=\sum_{0<\gamma\le T}S(T-\gamma)\,.$$

We shall show next the following theorem.

THEOREM 2. *For $T>T_o$, we have*

(i)

$$R_2(T)=O\left(T\frac{\log^2 T}{\log\log T}\right)$$

and

(ii) (on R.H.)

$$R_2(T)=O(T\log T)\,.$$

We should compare this with the following mean value theorem.

THEOREM 3. *For $T>T_o$, we have*

(i)

$$\int_0^T R_2(t)dt=O\left(T\left(\frac{\log T}{\log\log T}\right)^2\right)$$

and

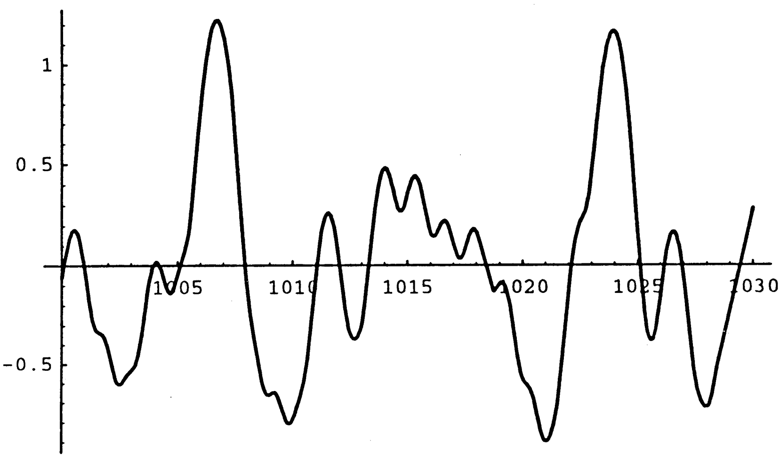
(ii) (on R.H.)

$$\int_0^T R_2(t)dt=-\left(\frac{1}{\pi}\int_{1/2}^{\infty}\log|\zeta(\sigma)|d\sigma\right)\cdot\frac{T}{2\pi}\log T-\frac{T}{2\pi^2}\log|\zeta(1+iT)|+O(T)\,.$$

It is well known that

$$\log|\zeta(1+iT)|=\Omega_{\pm}(\log\log\log T)\,.$$

In fact, the oscillating property of $\log|\zeta(1+iT)|$ is very interesting and we can



see it easily using "Mathematica" in the above graph of $\log|\zeta(1+iT)|$ for $1000 \leq T \leq 1030$, for example.

As a consequence of (ii) of Theorem 3, we get the following result which should be compared with the Nevalinna's result mentioned above.

COROLLARY 1 (On R.H.).

$$\int_1^T \frac{R_2(t)}{t} dt = -\left(\frac{1}{4\pi^2} \int_{1/2}^{\infty} \log|\zeta(\sigma)| d\sigma\right) \cdot \log^2 T + O(\log T).$$

To prove (i) of Theorem 2 and (i) of Theorem 3, we shall frequently use the following lemma which looks weak but the dependence on X works very well in the present situation.

LEMMA 1. *Suppose that* $1 < X \ll T^{8/7-\varepsilon}$, $\varepsilon > 0$. *Then we have*

$$\sum_{0<\gamma\leq T} X^{i\gamma} \ll T \log X + \min\left(\frac{\log T}{\log X}, T \log T\right).$$

On the other hand, to prove (ii) of Theorem 2 and (ii) of Theorem 3, we need the following finer lemma. Lemmas 1 and 2 have been proved in [14]. In fact, (ii) of Theorem 2 is stated and proved in Fujii [12], in another context.

LEMMA 2 (On R.H.). *For* $X > 1$ *and* $T > T_o$, *we have*

$$\sum_{\gamma\le T} X^{i\gamma} = -\frac{T}{2\pi}\frac{\Lambda(X)}{\sqrt{X}} + M(X, T) + X^{iT}S(T) + O(B(X, T))$$
$$+O\Bigg(\min\Bigg\{\sqrt{X}\log X\cdot\frac{\log T}{(\log\log T)^2}, \sqrt{X}\log(2X)$$
$$+X^{1/\log\log T}\frac{\log T}{\log((\log\frac{T}{X})+2)}$$
$$+\sqrt{X}\sqrt{\frac{\log T}{\log\log T}}\frac{1}{\log((\frac{X}{\log T}\log\log T)+2)}\Bigg\}\Bigg),$$

where we put

$$M(X, T) = \frac{1}{2\pi}\int_1^T X^{it}\log\frac{t}{2\pi}\,dt$$

and

$$B(X, T) = \frac{1}{\sqrt{X}}\sum_{\substack{X/2<k<2X\\ k\ne X}}\Lambda(k)\min\left(T, \frac{1}{|\log\frac{X}{k}|}\right)$$
$$= O\left(\frac{\log(2X)}{\sqrt{X}}\min\left(T, \frac{X}{|X-P(X)|}\right)\right) + O(\sqrt{X}\log(3X)\log\log(3X)),$$

$P(X)$ being the nearest prime power to X other than X itself.

Theorem 1 with (ii) of Theorem 2 implies the following consequence.

Corollary 2. *For any $T>T_o$, there exist γ and γ' (>0) such that*

$$|T-(\gamma+\gamma')| < \frac{C}{\log T},$$

where C is some positive constant.

In the present context, we need R.H. to get Corollary 2. However, as we have proved in [3], we do not need to assume R.H. to get it. In fact, we have proved it as an application of the author's mean value theorem on

$$\int_0^T (S(t+h)-S(t))^{2k}dt$$

for $h = \frac{C}{\log T}$.

For a comparison, it may be not worthless to recall that for any $T>T_o$, there exists a γ such that

$$|T-\gamma|<\frac{C}{\log\log\log T}$$

and that on R.H.,

$$|T-\gamma|<\frac{C}{\log\log T},$$

where C is some positive constant (cf. p. 224 of Titchmarsh [35]). However, in view of Montogomery's conjecture [28], it might be not correct to state that for any $T>T_o$, there exists a γ such that

$$|T-\gamma|<\frac{C}{\log T}$$

with some positive constant C.

If everything goes parallel, then it might be not correct to state that for any $T>T_o$, there exists γ and γ' (>0) such that

$$|T-(\gamma+\gamma')|<\frac{C}{T\log^2 T}$$

with some positive constant C. We might conjecture that

$$\overline{\lim_{n\to\infty}}\,(a_{n+1}-a_n)\frac{a_n\log^2 a_n}{4\pi^2}=+\infty$$

and

$$\underline{\lim_{n\to\infty}}\,(a_{n+1}-a_n)\frac{a_n\log^2 a_n}{4\pi^2}=0\,,$$

where a_n is written as the sum $\gamma+\gamma'$, $\gamma, \gamma'>0$ and we suppose that $a_n<a_{n+1}$ for $n\geq 1$.

As will be seen below, for example, in Theorem 6, that it is more comprehensive to treat the number

$$N(T, Y)=\#\{0<\gamma+\gamma'\leq Y;\, 0<\gamma, \gamma'\leq T\}$$

for $T\leq Y\leq 2T$. We shall only describe the result, since it is rather complicated.

THEOREM 1′. *For $T>T_o$, we have*

$$
\begin{aligned}
N(T, Y) = {} & N(Y-T)N(T) + L(Y-T)S(T) - L(T)S(Y-T) \\
& + \frac{1}{4\pi^2} YT \log(Y-T) \cdot \log T - \frac{1}{4\pi^2}(1+\log(2\pi)) YT \log(Y-T) \\
& - \frac{1}{8\pi^2} T^2 \log(Y-T) - \frac{1}{8\pi^2} Y^2 \log T \cdot \log(Y-T) \\
& - \frac{1}{8\pi^2} Y^2 \log Y \cdot \log(Y-T) + \frac{1}{8\pi^2} Y^2 \log T \cdot \log Y \\
& + \frac{1}{8\pi^2} T^2 \log T - \frac{1}{4\pi^2} YT \log T(2+\log(2\pi)) \\
& + \frac{1}{4\pi^2}\left(\frac{7}{2} + 3\log(2\pi) + \log^2(2\pi)\right) YT \\
& - \frac{1}{8\pi^2} Y^2 \left(\frac{7}{2} + 3\log(2\pi) + \log^2(2\pi)\right) \\
& + \frac{1}{4\pi^2}\left(\frac{3}{2} + \log(2\pi)\right) Y^2 \log(Y-T) - \frac{1}{8\pi^2} Y^2 \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{T}{Y}\right)^n \\
& + \frac{1}{8\pi^2} Y^2 \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\frac{Y-T}{Y}\right)^n + O\left(T \frac{\log^2 T}{\log\log T}\right).
\end{aligned}
$$

Under the Riemann Hypothesis, the remainder term $O(T\frac{\log^2 T}{\log\log T})$ becomes

$$O(T \log T).$$

We turn our attentions to an analogue of **(B)**. We first notice the simplest analogue of **(B)**.

THEOREM 4. *For any $X > 1$ and for $T > T_0$, we have*

$$\sum_{\substack{0 < \gamma, \gamma' \leq T \\ \gamma + \gamma' \leq T}} X^{\rho + \rho'} = \frac{T^2}{8\pi^2} \Lambda(X)^2 + O(T \log^2 T).$$

Theorems 1, 2 and 4 imply the following

COROLLARY 3. *The sequence $\gamma + \gamma'$ is uniformly distributed mod one, where $\gamma + \gamma'$ are arranged in the order of their magnitudes and $\gamma, \gamma' > 0$.*

We understand that the "multiplicity" of $\gamma + \gamma'$ is at least 2 for $\gamma \neq \gamma'$. So the above arrangement is with the "multiplicities". We expect that the "multiplicity" of $\gamma + \gamma'$ for $\gamma \neq \gamma'$ is exactly 2 as we have already stated above.

If we assume the Riemann Hypothesis, then we can obtain the second main term of the formula in Thoerem 4 as follows.

THEOREM 5 (On R.H.). *For $X>1$ and for $T>T_o$, we have*

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma+\gamma'\le T}} X^{i(\gamma+\gamma')}=\frac{1}{8\pi^2}\frac{\Lambda(X)^2}{X}T^2+\frac{X^{iT}}{4\pi^2 i\log X}T\log^2 T$$
$$+O\left(T\left(\frac{\log T}{\log\log T}\right)^2\sqrt{X}\log(3X)\right)+O\left(T\log T\left(\sqrt{X}\log(3X)\log\log(3X)\right.\right.$$
$$\left.\left.+\frac{1}{\log X}+\frac{\log(3X)}{\sqrt{X}}\min\left(T,\frac{X}{|X-P(X)|}\right)\right)\right)+O\left(\left(\frac{\log T}{\log X}\right)^2\right),$$

where $P(X)$ is defined in Lemma 2.

The dependence on X can be refined, although we shall not dare to do, if we apply our Lemma 2 fully, as can be seen in the proof of Theorem 5 given in the section 7 below.

If we ignore the dependence on X, the description becomes much simpler. Namely, we have the following.

COROLLARY 4 (On R.H.). *For any $X>1$ and for $T>T_o$, we have*

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma+\gamma'\le T}} X^{i(\gamma+\gamma')}=\frac{1}{8\pi^2}\frac{\Lambda(X)^2}{X}T^2+\frac{X^{iT}}{4\pi^2 i\log X}T\log^2 T+O\left(T\left(\frac{\log T}{\log\log T}\right)^2\right).$$

This is an analogue of (**B′**) and should be compared with the following simple application of (**B′**).

$$\sum_{0<\gamma,\gamma'\le T} X^{i(\gamma+\gamma')}=\left(\sum_{0<\gamma\le T} X^{i\gamma}\right)^2=\frac{1}{8\pi^2}\frac{\Lambda(X)^2}{X}T^2$$
$$-\frac{\Lambda(X)}{\sqrt{X}}\frac{X^{iT}}{2\pi^2 i\log X}T\log\frac{T}{2\pi}+O\left(\frac{\Lambda(X)}{\sqrt{X}}T\frac{\log T}{\log\log T}\right)+O(\log^2 T).$$

The difference is clear. Namely, when X is a prime power, then

$$\sum_{0<\gamma,\gamma'\le T} X^{i(\gamma+\gamma')}=O(\log^2 T),$$

while

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma+\gamma'\le T}} X^{i(\gamma+\gamma')}=\frac{X^{iT}}{4\pi^2 i\log X}T\log^2 T+O\left(T\left(\frac{\log T}{\log\log T}\right)^2\right).$$

We turn our attentions to give the analogues of (**C**), (**C′**) and (**D**).

THEOREM 6 (On R.H.). *Let α be a positive number. Let K be an integer ≥ 2. Then we have*

(i)

$$\lim_{T\to\infty}\frac{1}{\left(\frac{T}{2\pi}\right)^{3/2}\log T}\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma+\gamma'\le T}} e\left(\frac{\gamma+\gamma'}{2\pi K}\log\frac{\gamma+\gamma'}{2\pi e\alpha K}\right)$$

$$=\begin{cases}\frac{1}{\varphi(q)}\frac{\sqrt{2}}{1-i}\frac{2}{3K^{3/2}}\sum^{*}_{\chi:q}\chi(a)\bar{\tau}(\chi) & \text{if } \alpha=\frac{a}{q},\ (a,q)=1,\ a,q\ge 1\\ 0 & \text{if } \alpha \text{ is irrational}\end{cases}$$

and

(ii)

$$\lim_{T\to\infty}\frac{1}{\left(\frac{T}{2\pi}\right)^{3/2}\log T}\sum_{0<\gamma,\gamma'\le T} e\left(\frac{\gamma+\gamma'}{2\pi K}\log\frac{\gamma+\gamma'}{2\pi e\alpha K}\right)$$

$$=\begin{cases}\frac{1}{\varphi(q)}\frac{\sqrt{2}}{1-i}\frac{2}{3K^{3/2}}4(\sqrt{2}-1)\sum^{*}_{\chi:q}\chi(a)\bar{\tau}(\chi) & \text{if } \alpha=\frac{a}{q},\ (a,q)=1,\ a,q\ge 1\\ 0 & \text{if } \alpha \text{ is irrational},\end{cases}$$

where $*$ indicates that we sum over all Dirichlet characters χ mod q satisfying $\chi^K=\chi_o$, χ_o being the principal character mod q and we put

$$\tau(\chi)=\sum_{\substack{b=1\\(b,q)=1}}^{q}\chi(b)e\left(\frac{b}{q}\right).$$

Thus we see that the distribution of $\gamma+\gamma'$ and the rational numbers are connected with the aid of the above type of the exponential sums. In fact, these come from the following more general result.

THEOREM 7 (On R.H.). *If $0<b\le 2$, $T_o<T\le Y\le 2T$, then we have for any positive α*

$$\sum_{\substack{0<\gamma,\gamma'<T\\ \gamma+\gamma'<Y}} e^{ib(\gamma+\gamma')\log((\gamma+\gamma')/2\pi e\alpha)}$$

$$=\sqrt{\frac{\alpha^3}{b}}e^{\pi i/4}\sum_{k\le(T/2\pi\alpha)^b}\frac{\Lambda^2(k)k^{3/2b}}{k}e^{-2\pi ib\alpha k^{1/b}}$$

$$+\frac{T}{\pi}\sqrt{\frac{\alpha}{b}}e^{\pi i/4}\sum_{(T/2\pi\alpha)^b<k\le(Y/2\pi\alpha)^b}\frac{\Lambda^2(k)k^{1/2b}}{k}e^{-2\pi ib\alpha k^{1/b}}$$

$$-\sqrt{\frac{\alpha^3}{b}}e^{\pi i/4}\sum_{(T/2\pi\alpha)^b<k\le(Y/2\pi\alpha)^b}\frac{\Lambda^2(k)k^{3/2b}}{k}e^{-2\pi ib\alpha k^{1/b}}$$

$$+O(T^{1+b/2}\log^3 T\cdot\log\log T)+O(T^{(3-b)/2}\log^3 T).$$

We shall prove this theorem with explicit dependence on α (cf. The concluding remark 15-2 in the section 15 and Theorems 9 and 10 which will be proved from the section 8 till the section 12 below.) We may describe Theorem 7 in the form of **(c)** in **(I)**, stated at the beginning, as follows for $X \gg 1$ and for any positive α.

$$\sum_{n \le X} \Lambda^2(n) n^2 \cdot e(-\alpha n^2)$$
$$= \frac{e^{-\pi i/4}}{4\alpha^{3/2}} \sum_{\substack{0<\gamma,\gamma'<4\pi\alpha X^2 \\ \gamma+\gamma'<4\pi\alpha X^2}} e^{i(\gamma+\gamma')/2 \cdot \log((\gamma+\gamma')/2 \cdot 1/2\pi e\alpha)} + O(X^{5/2} \log^3 X \cdot \log\log X)\,.$$

This comes from Theorem 7 with $b=1/2$. This should be compared with the following result which is **(C″)** with $b=1/2$.

$$\sum_{n \le X} \Lambda(n)\sqrt{n} \cdot e(-\alpha n^2)$$
$$= -\frac{e^{-\pi i/4}}{2\sqrt{\alpha}} \sum_{0<\gamma<4\pi\alpha X^2} e^{i(\gamma/2)\log(\gamma/2 \cdot 1/2\pi e\alpha)} + O\left(\sqrt{X}\left(\frac{\log X}{\log\log X}\right)^2\right).$$

On the other hand, Theorem 7 with $b=1$ gives us the following result whose remainder term is slightly larger than what we should expect, unfortunately.

$$\sum_{n \le X} \Lambda^2(n)\sqrt{n} \cdot e(-\alpha n)$$
$$= \frac{e^{-\pi i/4}}{\alpha^{3/2}} \sum_{\substack{0<\gamma,\gamma'<2\pi\alpha X \\ \gamma+\gamma'<2\pi\alpha X}} e^{i(\gamma+\gamma')\log((\gamma+\gamma')/2\pi e\alpha)} + O(X^{3/2} \log^3 X \cdot \log\log X)\,.$$

However, this implies the following estimate for any positive α,

$$\sum_{\substack{0<\gamma,\gamma'<T \\ \gamma+\gamma'<T}} e^{i(\gamma+\gamma')\log((\gamma+\gamma')/2\pi e\alpha)} \ll T^{3/2} \log^3 T \cdot \log\log T\,,$$

which is not trivial in view of Theorems 1 and 2.

As another consequence of Theorem 7, we shall show also the analogue of **(D)** in the following form.

THEOREM 8 (On R.H.). *Let q be an integer ≥ 3 and K be an integer ≥ 2. Then G.R.H. for $L(s,\chi^K)$ for all Dirichlet character χ mod q is equivalent to the relation*

$$\sum_{\substack{0<\gamma,\gamma'\le T \\ \gamma+\gamma'\le T}} e\left(\frac{\gamma+\gamma'}{2\pi K} \log \frac{\gamma+\gamma'}{2\pi e \frac{a}{q} K}\right)$$

$$= \frac{1}{\varphi(q)} \frac{\sqrt{2}}{1-i} \frac{2}{3K^{3/2}} \left(\frac{T}{2\pi}\right)^{3/2} \left(\log \frac{T}{2\pi \frac{a}{q} K} - \frac{2}{3}\right) \sum_{\chi:q}^{*} \chi(a)\bar{\tau}(\chi) + O(T^{3/2-1/2K+\varepsilon})$$

for any $\varepsilon>0$ *and for any integer* a *with* $1\leq a\leq q$, $(a, q)=1$.

We expect that Theorems 6 and 8 are correct also for $K=1$. We may notice here that the condition $K\geq 2$ is much better than the previous one announced in Fujii [13].

We shall assume the Riemann Hypothesis from the section eight till the section thirteen. In other parts of this article, we shall not assume the Riemann Hypothesis unless we mention it.

Finally, we notice some notations. S_j's in the sections 2, 3, 4, 5, 6 and 7 are mutually independent in each section. However in the part from the section 8 till the section 11, S_j's are used in common and independently from the other part which consists of the sections 2, 3, 4, 5, 6 and 7. The author hopes that these will not course any confusion. C_1 and C_2 denote some absolute constants. C denotes some appropriate positive constant. $S_1(T)$ will be defined in the section 2.

§2. Proof of Theorem 1

We put $L(T)=\frac{1}{\pi}\vartheta(T)+1$, for simplicity. Then by the Riemann-von Mangoldt formula for $N(T)$, we get

$$\begin{aligned}\sum_{\substack{0<\gamma+\gamma'\leq T\\ 0<\gamma,\gamma'\leq T}}\cdot 1&=\sum_{0<\gamma\leq T}\ \sum_{0<\gamma'\leq T-\gamma}\cdot 1\\ &=\sum_{0<\gamma\leq T}L(T-\gamma)+\sum_{0<\gamma\leq T}S(T-\gamma)\\ &=S_1+S_2\,,\quad \text{say}\,.\end{aligned}$$

Using the Riemann-von Mangoldt formula again, we get

$$\begin{aligned}S_1&=\int_C^{T-C}L(T-t)d(L(t)+S(t))+\sum_{T-C<\gamma\leq T}L(T-\gamma)\\ &=(S_3+S_4)+S_5\,,\quad \text{say}\,.\end{aligned}$$

It is clear that

$$S_5\ll L(C)\log T\ll\log T\,.$$

And that

$$\begin{aligned}S_4&=[L(T-t)S(t)]_C^{T-C}+\int_C^{T-C}L'(T-t)S(t)dt\\ &=L(C)S(T-C)-L(T-C)S(C)+\int_C^{T-C}L'(T-t)S(t)dt\,.\end{aligned}$$

The last integral is

$$=[L'(T-t)S_1(t)]_C^{T-C}+\int_C^{T-C} L''(T-t)S_1(t)dt\,,$$

where we put

$$S_1(T)=\int_0^T S(t)dt$$

and shall use the estimate

$$S_1(T)\ll\log T\,.$$

As a result, we get

$$S_4=-L(T)S(C)+O(\log^2 T)\,.$$

Finally,

$$\begin{aligned}S_3&=\int_C^{T-C} L(T-t)L'(t)dt\\&=\int_C^{T-C} L(T-t)\frac{1}{2\pi}\log\frac{t}{2\pi}dt+\int_C^{T-C} L(T-t)\left(L'(t)-\frac{1}{2\pi}\log\frac{t}{2\pi}\right)dt\\&=S_6+S_7\,,\quad\text{say}\,.\end{aligned}$$

We get easily

$$\begin{aligned}S_7&=\int_C^{T-C} L(T-t)\left(\frac{1}{48\pi}\left(\frac{1}{t}\right)'+\frac{7}{5760\pi}\left(\frac{1}{t^3}\right)'+\cdots\right)dt\\&=-L(T)C_1+O(\log^2 T)\end{aligned}$$

and

$$\begin{aligned}S_6&=\int_C^{T-C} L(t)\frac{1}{2\pi}\log\frac{T-t}{2\pi}dt\\&=\int_C^{T-C}\left\{\frac{1}{2\pi}t\log t-\frac{1+\log(2\pi)}{2\pi}t+\frac{7}{8}+O\left(\frac{1}{t}\right)\right\}\frac{1}{2\pi}\log\frac{T-t}{2\pi}dt\\&=S_8+S_9+S_{10}+S_{11}\,,\quad\text{say}\,.\end{aligned}$$

S_8, S_9, S_{10} and S_{11} are easily evaluated as follows.

$$\begin{aligned}S_8&=\frac{1}{4\pi^2}\frac{(T-C)^2}{2}\log(T-C)\cdot\log\frac{C}{2\pi}-\frac{1}{8\pi^2}\int_C^{T-C} t\log\frac{T-t}{2\pi}dt\\&\quad+\frac{1}{8\pi^2}\int_C^{T-C} t^2\log t\cdot\frac{1}{T-t}dt+O(\log T)\\&=\frac{1}{4\pi^2}\frac{(T-C)^2}{2}\log(T-C)\cdot\log\frac{C}{2\pi}+S_{13}+S_{14}+O(\log T)\,,\quad\text{say}\,.\end{aligned}$$

$$S_{13}=-\frac{1}{8\pi^2}\left\{\frac{(T-C)^2}{2}\log\frac{C}{2\pi}+\frac{1}{2}\int_C^{T-C}\frac{t^2}{T-t}dt+O(\log T)\right\}$$

$$=-\frac{1}{8\pi^2}\left\{\frac{(T-C)^2}{2}\log\frac{C}{2\pi}+\frac{1}{2}(T^2(\log(T-C)-\log C)-2T(T-2C)\right.$$

$$\left.+\frac{1}{2}(T-C)^2)+O(\log T)\right\}$$

$$=-\frac{1}{8\pi^2}\left\{\frac{T^2}{2}\log\frac{T}{2\pi}-\frac{3}{4}T^2+C_1T+O(\log T)\right\}.$$

$$S_{14}=\frac{1}{8\pi^2}\frac{C}{T}\sum_{n=0}^{\infty}\frac{1}{T^n}\int_C^{T-C}t^{n+2}\log t\,dt$$

$$=\frac{1}{8\pi^2}\frac{1}{T}\sum_{n=0}^{\infty}\frac{1}{T^n}\left\{\frac{(T-C)^{n+3}}{n+3}\log(T-C)-\frac{(T-C)^{n+3}}{(n+3)^2}+O(1)\right\}$$

$$=\frac{1}{8\pi^2}\left\{T^2\log(T-C)\left(\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\frac{C}{T}\right)^n-\left(1-\frac{C}{T}\right)-\frac{1}{2}\left(1-\frac{C}{T}\right)^2\right)\right.$$

$$\left.-T^2\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\frac{C}{T}\right)^n-\left(1-\frac{C}{T}\right)-\frac{1}{4}\left(1-\frac{C}{T}\right)^2\right)\right\}+O\left(\frac{1}{T}\right)$$

$$=\frac{1}{8\pi^2}\left\{T^2\log^2 T-T^2\log T\cdot\log C-\frac{3}{2}T^2\log T+\frac{5}{4}T^2\right.$$

$$\left.-T^2\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\frac{C}{T}\right)^n+C_1T\log T+C_2T+O(\log T)\right\}.$$

Hence, we get

$$S_8=\frac{1}{8\pi^2}T^2\log T\cdot\log\frac{C}{2\pi}-\frac{1}{8\pi^2}\left\{\frac{T^2}{2}\log\frac{T}{2\pi}-\frac{3}{4}T^2+C_1T\right\}$$

$$+\frac{1}{8\pi^2}\left\{T^2\log^2 T-T^2\log T\cdot\log C-\frac{3}{2}T^2\log T+\frac{5}{4}T^2\right.$$

$$\left.-T^2\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\frac{C}{T}\right)^n\right\}+C_1T\log T+C_2T+O(\log T).$$

$$S_9=-\frac{1+\log(2\pi)}{4\pi^2}\left\{\frac{T^2}{2}\log\frac{C}{2\pi}+C_1T+O(\log T)+\frac{1}{2}\int_C^{T-C}\frac{t^2}{T-t}dt\right\}.$$

The last integral is

$$= -\frac{T^2}{2}\log\frac{C}{T} - \frac{3}{4}T^2 + C_1 T + O(1)\,.$$

Hence, we get

$$S_9 = -\frac{1+\log(2\pi)}{4\pi^2}\left\{\frac{T^2}{2}\log\frac{T}{2\pi} - \frac{3}{4}T^2 + C_1 T + O(\log T)\right\}.$$

Since

$$S_{10} + S_{11} = C_1 T\log T + C_2 T + O(\log^2 T)\,,$$

we get

$$\begin{aligned} S_1 = {} & \frac{1}{8\pi^2}T^2\log T\cdot\log\frac{C}{2\pi} - \frac{1}{8\pi^2}\left\{\frac{T^2}{2}\log\frac{T}{2\pi} - \frac{3}{4}T^2\right\} \\ & + \frac{1}{8\pi^2}\left\{T^2\log^2 T - T^2\log T\cdot\log C - \frac{3}{2}T^2\log T + \frac{5}{4}T^2\right. \\ & \left. - T^2\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\frac{C}{T}\right)^n\right\} - \frac{1+\log(2\pi)}{4\pi^2}\left\{\frac{T^2}{2}\log\frac{T}{2\pi} - \frac{3}{4}T^2\right\} \\ & + C_1 T\log T + C_2 T + O(\log^2 T) \\ = {} & \frac{1}{8\pi^2}T^2\log^2 T - \frac{1}{8\pi^2}T^2\log T\{3+2\log(2\pi)\} \\ & + \frac{1}{16\pi^2}T^2\{7+6\log(2\pi)+2\log^2(2\pi)-2\zeta(2)\} \\ & + C_1 T\log T + C_2 T + O(\log^2 T)\,. \end{aligned}$$

This proves our Theorem 1.

§3. Proof of Theorem 2

We shall first estimate $R_2(T)$ without assuming any unproved hypothesis. For this purpose, we shall use the following explicit formula for $S(T)$.

LEMMA 3 (p. 250 of Selberg [33]). *For* $2 \le X \le t^2$, $t \ge 2$, *we have*

$$\begin{aligned} S(t) = {} & \Im\left\{\frac{1}{\pi}\sum_{p<X^3}\frac{1}{p^{1/2+it}}\right\} + O\left(\left|\sum_{p<X^3}\frac{\Lambda(p)-\Lambda_X(p)}{\sqrt{p}\,\log p\cdot p^{it}}\right|\right) \\ & + O\left(\left|\sum_{p<X^{3/2}}\frac{\Lambda_X(p^2)}{p\log p\cdot p^{2it}}\right|\right) + O\left(\left(\sigma_{X,t}-\frac{1}{2}\right)\log T\right) \\ & + O\left(\left(\sigma_{X,t}-\frac{1}{2}\right)X^{(\sigma_{X,t}-1/2)}\int_{1/2}^{\infty}X^{1/2-\sigma}\left|\sum_{p<X^3}\frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+it}}\right|d\sigma\right), \end{aligned}$$

where we put

$$\Lambda_X(n)=\begin{cases}\Lambda(n) & \text{for } 1\le n\le X\\ \Lambda(n)\dfrac{(\log(X^3/n))^2-2(\log(X^2/n))^2}{2(\log X)^2} & \text{for } X\le n\le X^2\\ \Lambda(n)\dfrac{(\log(X^3/n))^2}{2(\log X)^2} & \text{for } X^2\le n\le X^3\end{cases}$$

and

$$\sigma_{X,t}=\frac{1}{2}+2\max_{\varrho}\left(\left|\beta-\frac{1}{2}\right|,\frac{2}{\log X}\right),$$

ϱ running here through all zeros $\beta+i\gamma$ of $\zeta(s)$ for which

$$|t-\gamma|\le\frac{X^{3|\beta-1/2|}}{\log X}.$$

We put $X=\sqrt{\log T}$ and $T_1=T-\sqrt{X}$. Now

$$\begin{aligned}R_2(T)&=\sum_{0<\gamma\le T_1}S(T-\gamma)+\sum_{T_1<\gamma\le T}S(T-\gamma)\\&=S_1+S_2,\quad\text{say}.\end{aligned}$$

Using the estimate $S(T-\gamma)\ll\log T$, we get simply,

$$S_2\ll\log T\sum_{T_1<\gamma\le T}\cdot 1\ll\sqrt{X}\log^2 T\ll\log^{5/2}T.$$

Using the above Lemma 3, we get

$$\begin{aligned}S_1=&\Im\left\{\frac{1}{\pi}\sum_{p<X^3}\frac{1}{p^{1/2+iT}}\sum_{0<\gamma\le T_1}e^{i\gamma\log p}\right\}\\&+O\left(\sqrt{T\log T}\left(\sum_{0<\gamma\le T_1}\left|\sum_{p<X^3}\frac{\Lambda(p)-\Lambda_X(p)}{\sqrt{p}\log p\cdot p^{i(T-\gamma)}}\right|^2\right)^{1/2}\right)\\&+O\left(\sqrt{T\log T}\left(\sum_{0<\gamma\le T_1}\left|\sum_{p<X^{3/2}}\frac{\Lambda_X(p^2)}{p\log p\cdot p^{2i(T-\gamma)}}\right|^2\right)^{1/2}\right)\\&+O\left(\sum_{0<\gamma\le T_1}\left(\sigma_{X,(T-\gamma)}-\frac{1}{2}\right)\log T\right)\\&+O\left(\left(\sum_{0<\gamma\le T_1}\left(\sigma_{X,(T-\gamma)}-\frac{1}{2}\right)^2X^{2(\sigma_{X,(T-\gamma)}-1/2)}\right)^{1/2}\right.\\&\left.\cdot\left(\int_{1/2}^{\infty}X^{1/2-\sigma}d\sigma\cdot\int_{1/2}^{\infty}X^{1/2-\sigma}\sum_{0<\gamma\le T_1}\left|\sum_{p<X^3}\frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+i(T-\gamma)}}\right|^2d\sigma\right)^{1/2}\right)\\=&S_3+O(\sqrt{T\log T}\sqrt{S_4})+O(\sqrt{T\log T}\sqrt{S_5})+O(S_6)+O(\sqrt{S_7}\sqrt{S_8}),\quad\text{say}.\end{aligned}$$

Using Lemma 1 stated in the introduction, we get

$$S_3 \ll \sum_{p<X^3} \frac{1}{p^{1/2}} T \log p \ll TX^{3/2} \ll T \log^{3/4} T.$$

Obviously, we have

$$\begin{aligned} S_4 &= \sum_{0<\gamma\le T_1} \sum_{p,q<X^3} \frac{\Lambda(p)-\Lambda_X(p)}{\sqrt{p}\,\log p\cdot p^{i(T-\gamma)}} \frac{\Lambda(q)-\Lambda_X(q)}{\sqrt{q}\,\log q\cdot q^{-i(T-\gamma)}} \\ &\ll T\log T \sum_{p<X^3} \frac{(\Lambda(p)-\Lambda_X(p))^2}{p\log^2 p} \\ &\quad + \sum_{p<q<X^3} \frac{|\Lambda(p)-\Lambda_X(p)}{\sqrt{p}\,\log p} \frac{|\Lambda(q)-\Lambda_X(q)|}{\sqrt{q}\,\log q} \left| \sum_{0<\gamma\le T_1} \left(\frac{q}{p}\right)^{i\gamma} \right| \\ &= S_9 + S_{10}, \quad \text{say}. \end{aligned}$$

By the definition, we get

$$S_9 \ll T\log T \sum_{X<p<X^3} \frac{1}{p} \ll T\log T.$$

Using Lemma 1, we get

$$\begin{aligned} S_{10} &\ll \sum_{p<q<X^3} \frac{|\Lambda(p)-\Lambda_X(p)|}{\sqrt{p}\,\log p} \frac{|\Lambda(q)-\Lambda_X(q)|}{\sqrt{q}\,\log q} \left(T\log\frac{q}{p} + \frac{\log T}{\log\frac{q}{p}} \right) \\ &\ll T \sum_{p<q<X^3} \frac{\log q}{\sqrt{pq}} + \log T \sum_{p<q<X^3} \frac{1}{\sqrt{pq}\log\frac{q}{p}} \ll \frac{T\log^{3/2} T}{\log\log T}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} S_5 &\ll T\log T \sum_{p<X^{3/2}} \frac{\Lambda_X^2(p^2)}{p^2\log^2 p} + \sum_{p<q<X^{3/2}} \frac{\Lambda_X(p^2)}{p\log q} \frac{\Lambda_X(q^2)}{q\log q} \left| \sum_{0<\gamma\le T_1} \left(\frac{q^2}{p^2}\right)^{i\gamma} \right| \\ &\ll T\log T + \sum_{p<q<X^{3/2}} \frac{\Lambda_X(p^2)}{p\log p} \frac{\Lambda_X(q^2)}{q\log q} \left(T\log\frac{q}{p} + \frac{\log T}{\log\frac{q}{p}} \right) \\ &\ll T\log T. \end{aligned}$$

We shall next evaluate the following sum for $1 \le \xi \le X^2$.

$$\begin{aligned} S_{11} &\equiv \sum_{0<\gamma\le T_1} \left(\sigma_{X,(T-\gamma)} - \frac{1}{2}\right)^2 \xi^{(\sigma_{X,(T-\gamma)}-1/2)} \\ &= \sum_{\substack{0<\gamma\le T_1 \\ \sigma_{X,(T-\gamma)}-1/2>4/\log X}} \left(\sigma_{X,(T-\gamma)} - \frac{1}{2}\right)^2 \xi^{(\sigma_{X,(T-\gamma)}-1/2)} + O\left(\frac{T\log T}{\log^2 X}\right) \\ &= S_{12} + O\left(\frac{T\log T}{\log^2 X}\right), \quad \text{say}. \end{aligned}$$

By the definition of $\sigma_{x,t}$, we get simply

$$
\begin{aligned}
S_{12} &\ll \sum_{0<\gamma\le T_1} \sum_{\substack{|T-\gamma-\gamma'|<X^{3|\beta'-1/2|}/\log X \\ \beta'-1/2>2/\log X}} \left(\beta'-\frac{1}{2}\right)^2 \xi^{2(\beta'-1/2)} \\
&\ll \sum_{\substack{0<\gamma'\ll T \\ \beta'-1/2>2/\log X}} \left(\beta'-\frac{1}{2}\right)^2 \xi^{2(\beta'-1/2)} \sum_{\substack{0<\gamma\le T_1 \\ |T-\gamma-\gamma'|<X^{3|\beta'-1/2|}/\log X}} \cdot 1 \\
&\ll \sum_{\substack{0<\gamma'\ll T \\ \beta'-1/2>2/\log X}} \left(\beta'-\frac{1}{2}\right)^2 \xi^{2(\beta'-1/2)} \left(\frac{X^{3|\beta'-1/2|}}{\log X}\log T+\log T\right) \\
&\ll \sum_{\substack{0<\gamma'\ll T \\ \beta'-1/2>2/\log X}} \left(\beta'-\frac{1}{2}\right)^2 (X^3\xi^2)^{(\beta'-1/2)} \frac{\log T}{\log X} \\
&\quad + \sum_{\substack{0<\gamma'\ll T \\ \beta'-1/2>2/\log X}} \left(\beta'-\frac{1}{2}\right)^2 \xi^{2(\beta'-1/2)} \log T \\
&= S_{13}+S_{14}, \quad \text{say},
\end{aligned}
$$

where $\beta'+i\gamma'$ runs over the zeros of $\zeta(s)$.

We get further

$$
\begin{aligned}
S_{13} &\ll \frac{\log T}{\log X} \sum_{\substack{0<\gamma'\ll T \\ \beta'-1/2>2/\log X}} \int_{1/2}^{\beta'} \left(\log(X^3\xi^2)\left(\sigma-\frac{1}{2}\right)^2+2\left(\sigma-\frac{1}{2}\right)\right)(X^3\xi^2)^{(\sigma-1/2)}d\sigma \\
&\ll \frac{\log T}{\log X} \sum_{\substack{0<\gamma'\ll T \\ \beta'-1/2>2/\log X}} \left(\int_{1/2}^{1/2+2/\log X} + \int_{1/2+2/\log X}^{\beta'}\right)\left(\log(X^3\xi^2)\left(\sigma-\frac{1}{2}\right)^2\right. \\
&\quad \left. +2\left(\sigma-\frac{1}{2}\right)\right)(X^3\xi^2)^{(\sigma-1/2)}d\sigma \\
&= S_{15}+S_{16}, \quad \text{say}.
\end{aligned}
$$

To estimate the last two integrals, we use the following estimate due to p. 232 of Selberg [33]. For $1/2\le\sigma\le 1$, we have

$$
\begin{aligned}
N(\sigma, T) &\equiv \sharp\{\beta+i\gamma\,;\,\beta>\sigma,\ 0<\gamma\le T\} \\
&\ll T\log T\cdot e^{-C(\sigma-1/2)\log T}.
\end{aligned}
$$

We get first

$$
\begin{aligned}
S_{15} &\ll \frac{\log T}{\log X}\left(\log(X^3\xi^2)\frac{1}{\log^3 X}+\frac{1}{\log^2 X}\right)\cdot(X^3\xi^2)^{2/\log X} N\left(\frac{1}{2}+\frac{2}{\log X}, CT\right) \\
&\ll \frac{T\log^2 T}{\log^3 X} e^{-C\log T/\log X}.
\end{aligned}
$$

We get next

$$S_{16} \ll \frac{\log T}{\log X} \int_{1/2+2/\log X}^{1} \left(\log(X^3\xi^2)\left(\sigma - \frac{1}{2}\right)^2 + 2\left(\sigma - \frac{1}{2}\right)\right)(X^3\xi^2)^{(\sigma-1/2)} N(\sigma, CT) d\sigma$$

$$\ll \frac{T\log^2 T}{\log X} \int_{1/2+2/\log X}^{1} \left(\log(X^3\xi^2)\left(\sigma - \frac{1}{2}\right)^2 + 2\left(\sigma - \frac{1}{2}\right)\right)(X^3\xi^2)^{(\sigma-1/2)} T^{-C(\sigma-1/2)} d\sigma$$

$$\ll \frac{T\log T}{\log^2 X} e^{-C\log T/\log X} .$$

Hence, we get by our choice of X,

$$S_{13} \ll T .$$

Similarly, we get

$$S_{14} \ll T .$$

Consequently, we get

$$S_{11} \ll \frac{T\log T}{\log^2 X} .$$

Using this, we get first

$$S_6 \ll \frac{T\log^2 T}{\log\log T}.$$

We get next

$$\sqrt{S_7}\sqrt{S_8} \ll \sqrt{T\log T \cdot \log X}$$
$$\cdot \left(\int_{1/2}^{\infty} X^{1/2-\sigma} \sum_{0<\gamma\le T_1} \left| \sum_{p<X^3} \frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+i(T-\gamma)}\log^2 X} \right|^2 d\sigma \right)^{1/2}$$
$$= \sqrt{T\log T \cdot \log X} \left(\int_{1/2}^{\infty} X^{1/2-\sigma} \cdot S_{17} d\sigma \right)^{1/2}, \quad \text{say} .$$

Using Lemma 1, we get

$$\int_{1/2}^{\infty} X^{1/2-\sigma} \cdot S_{17} d\sigma$$

$$\ll T\log T \int_{1/2}^{\infty} X^{1/2-\sigma} \sum_{p<X^3} \frac{\Lambda_X^2(p)\log^2(Xp)}{p^{2\sigma}\log^4 X} d\sigma$$
$$+ \int_{1/2}^{\infty} X^{1/2-\sigma} \sum_{p<q<X^3} \frac{\Lambda_X(p)\log(Xp)\Lambda_X(q)\log(Xq)}{(pq)^\sigma \log^4 X} \left| \sum_{0<\gamma\le T_1} \left(\frac{q}{p}\right)^{i\gamma} \right| d\sigma$$

$$\ll T \log T \sum_{p<X^3} \frac{\Lambda_X^2(p)\log^2(Xp)}{p \log^4 X \log(Xp^2)}$$

$$+ \sum_{p<q<X^3} \frac{\Lambda_X(p)\log(Xp)\Lambda_X(q)\log(Xq)}{\sqrt{pq}\log^4 X \cdot \log(Xpq)} \left(T \log\frac{q}{p} + \frac{\log T}{\log\frac{q}{p}} \right)$$

$$\ll T \log T \frac{1}{\log X} + T \frac{X^3}{\log^2 X} + \log T \frac{X^3}{\log X}.$$

Hence, we get

$$\sqrt{S_7}\sqrt{S_8} \ll T \frac{\log^{5/4} T}{\sqrt{\log\log T}}.$$

Combining all of our estimates, we get

$$S_1 \ll T \frac{\log^2 T}{\log\log T}.$$

Consequently, we get

$$R_2(T) \ll \frac{\log^2 T}{\log\log T}.$$

This proves (i) of Theorem 2.

To get the estimate of $R_2(T)$ under R.H., we use the following explicit formula.

LEMMA 4 (p. 185 of Selberg [33]) (On R.H.). *For* $4 \le X \le t^2$, $t > 2$, *we have*

$$S(t) = -\frac{1}{\pi} \sum_{n<X^2} \frac{\Lambda_X(n)\sin(t\log n)}{n^{\sigma_1}\log n} + O\left(\frac{1}{\log X} \left| \sum_{n<X^2} \frac{\Lambda_X(n)}{n^{\sigma_1+it}} \right| \right) + O\left(\frac{\log t}{\log X} \right),$$

where we put

$$\Lambda_X(n) = \begin{cases} \Lambda(n) & \text{for} \quad 1 \le n \le X \\ \Lambda(n)\dfrac{\log(X^2/n)}{\log X} & \text{for} \quad X \le n \le X^2 \end{cases}$$

and

$$\sigma_1 = \frac{1}{2} + \frac{1}{\log X}.$$

Using Lemma 2, we get our Theorem by the same manner as in pp. 99–100 of Fujii [12].

This proves all of Theorem 3.

§4. Proof of (ii) of Theorem 3 and Corollary 1

We shall assume the Riemann Hypothesis in this section.

We notice first that

$$\int_0^T R_2(t)dt = \int_0^T \sum_{0<\gamma\le t} S(t-\gamma)dt = \sum_{0<\gamma\le T} \int_\gamma^T S(t-\gamma)dt$$
$$= \sum_{0<\gamma\le T} \int_0^{T-\gamma} S(t)dt$$
$$= \sum_{0<\gamma\le T} S_1(T-\gamma)\,.$$

We shall use the following explicit formula for $S_1(T)$.

LEMMA 5 (p. 185 and p. 246 of Selberg [33]) (On R.H.). *For* $4\le X\le t^2$, $t>2$, *we have*

$$S_1(t) = -\frac{1}{\pi}\int_{1/2}^{\infty} \log|\zeta(\sigma)|d\sigma + \frac{1}{\pi}\sum_{n<X^2} \frac{\Lambda_X(n)\cos(t\log n)}{n^{\sigma_1}\log^2 n}\left(1+\frac{\log n}{\log X}\right)$$
$$+O\left(\frac{1}{\log^2 X}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+it}}\right|\right) + O\left(\frac{\log t}{\log^2 X}\right),$$

where we put, as in Lemma 4 *in the previous section,*

$$\Lambda_X(n) = \begin{cases} \Lambda(n) & \textit{for} \quad 1\le n\le X \\ \Lambda(n)\dfrac{(\log X^2/n)}{\log X} & \textit{for} \quad X\le n\le X^2 \end{cases}$$

and

$$\sigma_1 = \frac{1}{2} + \frac{1}{\log X}\,.$$

We put $X=T^b$ with a sufficiently small positive b and $T_1 = T-\sqrt{X}$. Now

$$\sum_{0<\gamma\le T} S_1(T-\gamma) = \sum_{0<\gamma\le T_1} S_1(T-\gamma) + \sum_{T_1<\gamma\le T} S_1(T-\gamma)$$
$$= S_1 + S_2\,, \quad \text{say}\,.$$

We get immediately,

$$S_2 \ll \log T \sum_{T_1<\gamma\le T} \cdot 1 \ll \sqrt{X}\log^2 T \ll T^{b/2}\log^2 T\,.$$

Using the above Lemma 5, we get

$$S_1 = -\left(\frac{1}{\pi}\int_{1/2}^{\infty} \log|\zeta(\sigma)|d\sigma\right)\cdot \sum_{0<\gamma\le T_1} \cdot 1$$

$$+\frac{1}{\pi}\Re\left\{\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)\sum_{0<\gamma\le T_1} e^{i\gamma\log n}\right\}$$

$$+O\left(\frac{\sqrt{T\log T}}{\log^2 X}\left(\sum_{0<\gamma\le T_1}\left|\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+i(T-\gamma)}}\right|^2\right)^{1/2}\right)+O\left(\sum_{0<\gamma\le T_1}\frac{\log(T-\gamma)}{\log^2 X}\right)$$

$$=S_3+S_4+O\left(\frac{\sqrt{T\log T}}{\log^2 T}\sqrt{S_5}\right)+O\left(\frac{1}{\log^2 X}S_6\right), \quad \text{say}.$$

S_3 is clearly equal to

$$-\left(\frac{1}{\pi}\int_{1/2}^{\infty}\log|\zeta(\sigma)|d\sigma\right)\cdot\frac{T}{2\pi}\log T+O(T).$$

We get trivially,

$$\frac{1}{\log^2 X}S_6 \ll \frac{1}{\log^2 X}T\log^2 T \ll T.$$

Applying Lemma 2, we get clearly,

$$S_4=\frac{1}{\pi}\Re\left\{\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)\left(-\frac{T_1}{2\pi}\frac{\Lambda(n)}{\sqrt{n}}\right.\right.$$

$$+\frac{1}{2\pi}\int_1^{T_1} n^{it}\log\frac{t}{2\pi}dt+n^{iT_1}S(T_1)+O\left(\frac{1}{\sqrt{n}}\sum_{\substack{n/2<k<2n\\ k\ne n}}\Lambda(k)\frac{1}{|\log\frac{n}{k}|}\right)$$

$$\left.\left.+O\left(\sqrt{n}\log n\cdot\frac{\log T}{(\log\log T)^2}\right)\right)\right\}$$

$$=S_7+S_8+S_9+S_{10}+S_{11}, \quad \text{say}.$$

We get first

$$S_7=-\frac{T_1}{2\pi^2}\Re\left\{\sum_{n<X^2}\frac{\Lambda_X(n)\Lambda(n)}{n^{1/2+\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)\right\}.$$

We get next

$$S_8=\frac{1}{\pi}\Re\left\{\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)\frac{1}{2\pi}\int_1^{T_1}n^{it}\log\frac{t}{2\pi}dt\right\}$$

$$\ll \sum_{n<X^2}\frac{\Lambda(n)}{\sqrt{n}\log^3 n}\log T \ll \frac{X}{\log^3 X}\log T \ll T.$$

Using the stimate on $S(T)$ under R.H., we get

$$S_9=\frac{1}{\pi}\Re\left\{\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)n^{iT_1}S(T_1)\right\}$$
$$\ll\frac{X}{\log^2 X}\frac{\log T}{\log\log T}\ll T\,.$$

We get simply,

$$S_{10}=\frac{1}{\pi}\Re\left\{\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)O\left(\frac{1}{\sqrt{n}}\sum_{\substack{n/2<k<2n\\ k\neq n}}\Lambda(k)\frac{1}{|\log\frac{n}{k}|}\right)\right\}$$
$$\ll\sum_{n<X^2}\frac{\Lambda(n)}{\log^2 n}\sum_{\substack{n/2<k<2n\\ k\neq n}}\frac{\Lambda(k)}{|n-k|}\ll T$$

and

$$S_{11}=\frac{1}{\pi}\Re\left\{\sum_{n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)O\left(\sqrt{n}\,\log n\cdot\frac{\log T}{(\log\log T)^2}\right)\right\}$$
$$\ll\sum_{n<X^2}\frac{\Lambda_X(n)}{\log n}\frac{\log T}{(\log\log T)^2}\ll X^2\frac{\log T}{(\log\log T)^2}\ll T\,.$$

Thus we get

$$S_4=-\frac{T}{2\pi^2}\Re\left\{\sum_{n<X^2}\frac{\Lambda_X(n)\Lambda(n)}{n^{1/2+\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)\right\}+O(T)\,.$$

Finally, we shall evaluate S_5. Applying Lemma 2 directly, we get

$$S_5=\sum_{0<\gamma\leq T_1}\sum_{m,n<X^2}\frac{\Lambda_X(n)}{n^{\sigma_1+i(T-\gamma)}}\frac{\Lambda_X(m)}{m^{\sigma_1-i(T-\gamma)}}$$
$$\ll T\log T\sum_{n<X^2}\frac{\Lambda_X^2(n)}{n^{2\sigma_1}}+\sum_{m<n<X^2}\frac{\Lambda_X(n)\Lambda_X(m)}{(mn)^{\sigma_1}}\left|\sum_{0<\gamma\leq T_1}\left(\frac{n}{m}\right)^{i\gamma}\right|$$
$$\ll T\log T\cdot\log^2 X+\sum_{m<n<X^2}\frac{\Lambda_X(n)\Lambda_X(m)}{(mn)^{\sigma_1}}\left\{\frac{T}{2\pi}\frac{\Lambda\left(\frac{n}{m}\right)}{\sqrt{\frac{n}{m}}}+O\left(\frac{\log T}{\log\frac{n}{m}}\right)+\left(\frac{n}{m}\right)^{iT_1}S(T_1)\right.$$
$$\left.+O\left(\frac{1}{\sqrt{\frac{n}{m}}}\sum_{\substack{(n/m)/2<k<2(n/m)\\ k\neq n/m}}\Lambda(k)\frac{1}{|\log\frac{n}{mk}|}\right)+O\left(\sqrt{\frac{n}{m}}\log\frac{n}{m}\cdot\frac{\log T}{(\log\log T)^2}\right)\right\}$$
$$=T\log T\cdot\log^2 X+S_{12}+S_{13}+S_{14}+S_{15}+S_{16}\,,\quad\text{say}\,.$$

We get first

$$S_{12}=\sum_{m<n<X^2}\frac{\Lambda_X(n)\Lambda_X(m)}{(mn)^{\sigma_1}}\frac{T}{2\pi}\frac{\Lambda\left(\frac{n}{m}\right)}{\sqrt{\frac{n}{m}}}$$

$$\ll T\sum_{\substack{p^l<p^k<X^2\\ k\geq 2, l\leq k-1}}\frac{\log^2 p}{p^k}\Lambda(p^{k-l})$$

$$\ll T\sum_{p<X}\log^3 p\sum_{2\leq k\leq \log X/\log p}\frac{1}{p^k}\sum_{l\leq k-1}\cdot 1$$

$$\ll T\sum_{p<X}\frac{\log^3 p}{p^2}\sum_{2\leq k\leq\log X/\log p}k\ll T\log^2 X\sum_{p<X}\frac{\log p}{p^2}\ll T\log^2 X\,.$$

We get next

$$S_{13}\ll\sum_{m\leq n/2<X^2}\frac{\Lambda(n)\Lambda(m)}{\sqrt{mn}}\log T+\sum_{n<X^2}\Lambda(n)\sum_{n/2<m<n\ll X^2}\frac{\Lambda(m)}{n-m}\log T$$

$$\ll X^2\log T+\sum_{n<X^2}\Lambda(n)\log(3n)\cdot\log\log(3n)\cdot\log T$$

$$\ll X^2\log^2 T\cdot\log\log T\ll T\,,$$

$$S_{14}=\sum_{m<n<X^2}\frac{\Lambda_X(n)\Lambda_X(m)}{(mn)^{\sigma_1}}\left(\frac{n}{m}\right)^{iT_1}S(T_1)\ll X^2\frac{\log T}{\log\log T}\ll T$$

and

$$S_{16}\ll\sum_{m<n<X^2}\frac{\Lambda_X(n)\Lambda_X(m)}{(mn)^{\sigma_1}}\sqrt{\frac{n}{m}}\log\frac{n}{m}\cdot\frac{\log T}{(\log\log T)^2}$$

$$\ll\sum_{m<n<X^2}\frac{\Lambda(m)\Lambda(n)}{m}\left(\frac{\log T}{\log\log T}\right)^2\ll X^2\log X\left(\frac{\log T}{\log\log T}\right)^2\ll T\,.$$

We get clearly,

$$S_{15}\ll\sum_{m<n<X^2}\frac{\Lambda_X(n)\Lambda_X(m)}{(mn)^{\sigma_1}}\frac{1}{\sqrt{\frac{n}{m}}}\sum_{\substack{(n/m)/2<k<2(n/m)\\ k\neq n/m}}\Lambda(k)\frac{1}{\left|\log\frac{n}{mk}\right|}$$

$$\ll\sum_{d<2X^2}\frac{1}{d}\left(\sum_{mk=d}\Lambda(m)\Lambda(k)\right)\sum_{\substack{d/2<n<2d\\ n\neq d}}\frac{\Lambda(n)}{\left|\log\frac{n}{d}\right|}$$

$$\ll\sum_{d<2X^2}\log^3(3d)\cdot\log\log(3d)\ll X^2\log^2 X\cdot\log\log X\ll T\,.$$

Thus we get

$$S_5 \ll T \log T \cdot \log^2 X .$$

Consequently, we get

$$\begin{aligned}\int_0^T R_2(t)dt &= -\left(\frac{1}{\pi}\int_{1/2}^{\infty} \log|\zeta(\sigma)|d\sigma\right)\cdot\frac{T}{2\pi}\log T + O(T)\\ &\quad -\frac{T}{2\pi^2}\Re\left\{\sum_{n<X^2}\frac{\Lambda_X(n)\Lambda(n)}{n^{1/2+\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)\right\}\\ &\quad + O\left(\frac{\sqrt{T\log T}}{\log^2 X}\sqrt{T\log T\cdot\log^2 X}\right)\\ &= -\left(\frac{1}{\pi}\int_{1/2}^{\infty}\log|\zeta(\sigma)|d\sigma\right)\cdot\frac{T}{2\pi}\log T\\ &\quad -\frac{T}{2\pi^2}\Re\left\{\sum_{n<X^2}\frac{\Lambda_X(n)\Lambda(n)}{n^{1/2+\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log \mathrm{X}}\right)\right\} + O(T) .\end{aligned}$$

At this stage we should simplify the last sum. We put $\delta = \frac{1}{\log X}$. Then we have

$$\begin{aligned}&\sum_{n<X^2}\frac{\Lambda_X(n)\Lambda(n)}{n^{1/2+\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)\\ &= \sum_{n<X}\frac{\Lambda^2(n)}{n^{1+\delta+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right) + O\left(\sum_{X<n<X^2}\frac{\Lambda^2(n)}{n^{1+\delta+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)\right)\\ &= \sum_{p<X}\frac{1}{p^{1+\delta+iT}} + \sum_{p<X}\frac{1}{p^{1+\delta+iT}}\frac{\log p}{\log X} + O\left(\sum_{X<p<X^2}\frac{1}{p}\right) + O(1)\\ &= \sum_{p<X}\frac{1}{p^{1+\delta+iT}} + O(1)\\ &= \log\zeta(1+\delta+iT) + O(1)\\ &= \log\zeta(1+iT) + O(1) ,\end{aligned}$$

since by p. 135 of Titchmarsh [35], we get

$$\begin{aligned}\log\zeta(1+\delta+iT) - \log\zeta(1+iT) &= \int_1^{1+\delta}\frac{\zeta'}{\zeta}(\sigma+iT)d\sigma \ll \frac{(\log T)^{2/3}(\log\log T)^{1/3}}{\log X}\\ &\ll 1 .\end{aligned}$$

Thus we get

$$\Re\left\{\sum_{n<X^2}\frac{\Lambda_X(n)\Lambda(n)}{n^{1/2+\sigma_1+iT}\log^2 n}\left(1+\frac{\log n}{\log X}\right)\right\} = \log|\zeta(1+iT)| + O(1) .$$

This proves (ii) of Theorem 3.

We shall next prove Corollary 1.

Using the integrations by parts, we get by (ii) of Theorem 3,

$$\begin{aligned}
\int_1^T \frac{R_2(t)}{t}dt &= \int_1^T \left(\int_0^t R_2(t)dt\right)' \frac{1}{t}dt \\
&= \left[\frac{1}{t}\int_0^t R_2(t)dt\right]_1^T + \int_1^T \left(\int_0^t R_2(t)dt\right)\frac{1}{t^2}dt \\
&= -\left(\frac{1}{2\pi^2}\int_{1/2}^{\infty} \log|\zeta(\sigma)|d\sigma\right)\cdot \int_1^T \frac{\log t}{t}dt \\
&\quad -\frac{1}{2\pi^2}\int_1^T \frac{\log|\zeta(1+it)|}{t}dt + O(\log T) \\
&= -\left(\frac{1}{4\pi^2}\int_{1/2}^{\infty} \log|\zeta(\sigma)|d\sigma\right)\cdot \log^2 T \\
&\quad -\frac{1}{2\pi^2}\Re\left\{\int_1^T \frac{\log\zeta(1+it)}{t}dt\right\} + O(\log T).
\end{aligned}$$

The last integral is

$$\begin{aligned}
&= \int_1^T \frac{\log\zeta(1+\delta_1+it)}{t}dt + \int_1^T \frac{\log\zeta(1+it) - \log\zeta(1+\delta_1+it)}{t}dt \\
&= \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{1+\delta_1}\log n}\int_1^T \frac{1}{t}n^{-it}dt + O(\log T) \\
&= O(\log T),
\end{aligned}$$

where we put $\delta_1 = \frac{1}{\log T}$. Hence, we get

$$\int_1^T \frac{R_2(t)}{t}dt = -\left(\frac{1}{4\pi^2}\int_{1/2}^{\infty} \log|\zeta(\sigma)|d\sigma\right)\cdot \log^2 T + O(\log T).$$

This proves our Corollary 1.

§5. Proof of (i) of Theorem 3

We need first the following explicit formula for $S_1(T)$.

LEMMA 6 (p. 246 of Selberg [33]). *For* $2 \le X \le t^2$, $t \ge 2$, *we have*

$$S_1(t)=-\frac{1}{\pi}\int_{1/2}^{\infty}\log|\zeta(\sigma)|d\sigma$$
$$+\frac{1}{\pi}\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}}\log^2 n}\left(1+\left(\sigma_{X,t}-\frac{1}{2}\right)\log n\right)\cos(t\log n)$$
$$+O\left(\left(\sigma_{X,t}-\frac{1}{2}\right)^2\left|\sum_{n<X^3}\frac{\Lambda_X(n)}{n^{\sigma_{X,t}+it}}\right|\right)+O\left(\left(\sigma_{X,t}-\frac{1}{2}\right)^2\log t\right),$$

where $\Lambda_X(n)$ and $\sigma_{X,t}$ are the same as in Lemma 3 *in the section* 3.

From this we can deduce the following formula.

LEMMA 7. *For $2\le X\le t^2$, $2\le t\le T$, we have*

$$S_1(t)=-\frac{1}{\pi}\int_{1/2}^{\infty}\log|\zeta(\sigma)|d\sigma+\Re\left\{\frac{1}{\pi}\sum_{p<X^3}\frac{1}{p^{1/2+it}\log p}\right\}$$
$$+\Re\left\{\frac{1}{\pi}\sum_{p^r<X^3,r\ge 3}\frac{\Lambda_X(p^r)}{p^{r/2+irt}\log^2 p^r}\right\}+O\left(\left(\sigma_{X,t}-\frac{1}{2}\right)\left|\sum_{p<X^3}\frac{1}{p^{1/2+it}}\right|\right)$$
$$+O\left(\left|\sum_{p<X^3}\frac{\Lambda(p)-\Lambda_X(p)}{\sqrt{p}\log^2 p\cdot p^{it}}\right|\right)+O\left(\left(\sigma_{X,t}-\frac{1}{2}\right)\left|\sum_{p<X^3}\frac{\Lambda(p)-\Lambda_X(p)}{\sqrt{p}\log p\cdot p^{it}}\right|\right)$$
$$+O\left(\left|\sum_{p<X^{3/2}}\frac{\Lambda_X(p^2)}{p\log^2 p\cdot p^{2it}}\right|\right)$$
$$+O\left(\left(\sigma_{X,t}-\frac{1}{2}\right)^2 X^{(\sigma_{X,t}-1/2)}\int_{1/2}^{\infty}X^{1/2-\sigma}\left|\sum_{p<X^3}\frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+it}}\right|d\sigma\right)$$
$$+O\left(\left(\sigma_{X,t}-\frac{1}{2}\right)X^{(\sigma_{X,t}-1/2)}\int_{1/2}^{\infty}X^{1/2-\sigma}\left|\sum_{p<X^3}\frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+it}\log p}\right|d\sigma\right)$$
$$+O\left(\left(\sigma_{X,t}-\frac{1}{2}\right)^2\log T\right)+O\left(\left(\sigma_{X,t}-\frac{1}{2}\right)\log\log T\right).$$

Using this we shall evaluate our integral.

We put $X=\log^b T$ with a sufficiently small positive constant b and $T_1=T-\sqrt{X}$. We denote any sufficiently small positive number by b below in the present section. Now as in the previous section we decompose the integral as

$$\int_0^T R_2(t)=\sum_{0<\gamma\le T_1}S_1(T-\gamma)+\sum_{T_1<\gamma\le T}S_1(T-\gamma)$$
$$=S_1+S_2\,,\quad\text{say}\,.$$

Clearly, we get

$$S_2\ll\log T\sum_{T_1<\gamma\le T}\cdot 1\ll\sqrt{X}\log^2 T\,.$$

Using the above Lemma 7, we get

$$S_1=-\frac{1}{\pi}\frac{T}{2\pi}\log T\int_{1/2}^{\infty}\log|\zeta(\sigma)|\,d\sigma+O(T)$$

$$+\Re\left\{\frac{1}{\pi}\sum_{p<X^3}\frac{1}{p^{1/2+iT}\log p}\sum_{0<\gamma\le T_1}e^{i\gamma\log p}\right\}$$

$$+\Re\left\{\frac{1}{\pi}\sum_{p^r<X^3,r\ge 3}\frac{\Lambda_X(p^r)}{p^{r/2+irT}\log^2 p^r}\sum_{0<\gamma\le T_1}e^{i\gamma r\log p}\right\}$$

$$+O\left(\left(\sum_{0<\gamma\le T_1}\left(\sigma_{X,(T-\gamma)}-\frac{1}{2}\right)^2\right)^{1/2}\cdot\left(\sum_{0<\gamma\le T_1}\left|\sum_{p<X^3}\frac{1}{p^{1/2+i(T-\gamma)}}\right|^2\right)^{1/2}\right)$$

$$+O\left(\sqrt{T\log T}\left(\sum_{0<\gamma\le T_1}\left|\sum_{p<X^3}\frac{\Lambda(p)-\Lambda_X(p)}{\sqrt{p}\log^2 p\cdot p^{i(T-\gamma)}}\right|^2\right)^{1/2}\right)$$

$$+O\left(\left(\sum_{0<\gamma\le T_1}\left(\sigma_{X,(T-\gamma)}-\frac{1}{2}\right)^2\right)^{1/2}\cdot\left(\sum_{0<\gamma\le T_1}\left|\sum_{p<X^3}\frac{\Lambda(p)-\Lambda_X(p)}{\sqrt{p}\log p\cdot p^{i(T-\gamma)}}\right|^2\right)^{1/2}\right)$$

$$+O\left(\sqrt{T\log T}\left(\sum_{0<\gamma\le T_1}\left|\sum_{p<X^{3/2}}\frac{\Lambda_X(p^2)}{p\log^2 p\cdot p^{2i(T-\gamma)}}\right|^2\right)^{1/2}\right)$$

$$+O\left(\sum_{0<\gamma\le T_1}\left(\sigma_{X,(T-\gamma)}-\frac{1}{2}\right)^2\log T\right)+O\left(\sum_{0<\gamma\le T_1}\left(\sigma_{X,(T-\gamma)}-\frac{1}{2}\right)\log\log T\right)$$

$$+O\left(\left(\sum_{0<\gamma\le T_1}\left(\sigma_{X,(T-\gamma)}-\frac{1}{2}\right)^2 X^{2(\sigma_{X,(T-\gamma)}-1/2)}\right)^{1/2}\right.$$

$$\left.\cdot\left(\int_{1/2}^{\infty}X^{1/2-\sigma}d\sigma\cdot\int_{1/2}^{\infty}X^{1/2-\sigma}\sum_{0<\gamma\le T_1}\left|\sum_{p<X^3}\frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+i(T-\gamma)}\log p}\right|^2 d\sigma\right)^{1/2}\right)$$

$$+O\left(\left(\sum_{0<\gamma\le T_1}\left(\sigma_{X,(T-\gamma)}-\frac{1}{2}\right)^4 X^{2(\sigma_{X,(T-\gamma)}-1/2)}\right)^{1/2}\right.$$

$$\left.\cdot\left(\int_{1/2}^{\infty}X^{1/2-\sigma}d\sigma\cdot\int_{1/2}^{\infty}X^{1/2-\sigma}\sum_{0<\gamma\le T_1}\left|\sum_{p<X^3}\frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+i(T-\gamma)}}\right|^2 d\sigma\right)^{1/2}\right)$$

$$=-\frac{1}{\pi}\frac{T}{2\pi}\log T\int_{1/2}^{\infty}\log|\zeta(\sigma)|\,d\sigma+O(T)$$

$$+S_3+S_4+O(\sqrt{S_5}\sqrt{S_6})+O(\sqrt{T\log T}\sqrt{S_7})+O(\sqrt{S_5}\sqrt{S_8})$$

$$+O(\sqrt{T\log T}\sqrt{S_9})+O(S_5\cdot\log T)+O(S_{10}\cdot\log\log T)$$

$$+O(\sqrt{S_{11}}\sqrt{S_{12}})+O(\sqrt{S_{13}}\sqrt{S_{14}}),\quad\text{say}.$$

Using Lemma 1 stated in the introduction, we get

$$S_3\ll\sum_{p<X^3}\frac{1}{p^{1/2}\log p}T\log p\ll T\frac{X^{3/2}}{\log X}\ll T\log^b T$$

and

$$S_4 \ll T \sum_{p^r < X^3, r \geq 3} \frac{\log p^r}{p^{r/2} \log p^r} \ll T .$$

By the result on S_{11} in the section 3, we get

$$S_5 \ll \frac{T \log T}{\log^2 X} \ll \frac{T \log T}{(\log\log T)^2} .$$

Using Lemma 1, we get

$$\begin{aligned}
S_6 &\ll \sum_{0<\gamma\leq T_1} \sum_{p,q<X^3} \frac{1}{\sqrt{pq}} \left(\frac{q}{p}\right)^{i(T-\gamma)} \\
&\ll \sum_{0<\gamma\leq T_1} \sum_{p<X^3} \frac{1}{p} + \sum_{p<q<X^3} \frac{1}{\sqrt{pq}} \left| \sum_{0<\gamma\leq T_1} \left(\frac{q}{p}\right)^{i\gamma} \right| \\
&\ll T \log T \log\log X + \sum_{p<q<X^3} \frac{1}{\sqrt{pq}} \left(T \log \frac{q}{p} + \frac{\log T}{\log \frac{q}{p}} \right) \\
&\ll T \log T \log\log X + \frac{TX^3}{\log X} \ll T \log T \cdot \log\log\log T .
\end{aligned}$$

By the result on S_4 in the section 3, we get

$$S_8 \ll T \log T .$$

In the same manner, we get

$$\begin{aligned}
S_7 &= \sum_{0<\gamma\leq T_1} \sum_{p,q<X^3} \frac{\Lambda(p)-\Lambda_X(p)}{\sqrt{p} \log^2 p \cdot p^{i(T-\gamma)}} \frac{\Lambda(q)-\Lambda_X(q)}{\sqrt{q} \log^2 q \cdot q^{-i(T-\gamma)}} \\
&\ll T \log T \sum_{p<X^3} \frac{(\Lambda(p)-\Lambda_X(p))^2}{p \log^4 p} \\
&\quad + \sum_{p<q<X^3} \frac{|\Lambda(p)-\Lambda_X(p)|}{\sqrt{p} \log^2 p} \frac{|\Lambda(q)-\Lambda_X(q)|}{\sqrt{q} \log^2 q} \left| \sum_{0<\gamma\leq T_1} \left(\frac{q}{p}\right)^{i\gamma} \right| \\
&= S_{15} + S_{16} , \quad \text{say} .
\end{aligned}$$

By the definition, we get

$$S_{15} \ll T \log T \sum_{X<p<X^3} \frac{1}{p \log^2 p} \ll T \frac{\log T}{\log^2 X} \ll \frac{T \log T}{(\log\log T)^2} .$$

Using Lemma 1, we get

$$S_{16} \ll \sum_{p<q<X^3} \frac{|\Lambda(p)-\Lambda_X(p)|}{\sqrt{p}\log^2 p} \frac{|\Lambda(q)-\Lambda_X(q)|}{\sqrt{q}\log^2 q}\left(T\log\frac{q}{p}+\frac{\log T}{\log\frac{q}{p}}\right)$$

$$\ll T\sum_{p<q<X^3}\frac{1}{\sqrt{pq}\log p}+\log T\sum_{p<q<X^3}\frac{1}{\sqrt{pq}\log\frac{q}{p}\log^2 p\log^2 q}$$

$$\ll T\log^b T.$$

Hence, we get

$$S_7 \ll \frac{T\log T}{(\log\log T)^2}.$$

Similarly, we get

$$S_9 \ll T\log T\sum_{p<X^{3/2}}\frac{\Lambda_X^2(p^2)}{p^2\log^4 p}+\sum_{p<q<X^{3/2}}\frac{\Lambda_X(p^2)}{p\log^2 p}\frac{\Lambda_X(q^2)}{q\log^2 q}\left|\sum_{0<\gamma\le T_1}\left(\frac{q^2}{p^2}\right)^{i\gamma}\right|$$

$$\ll T\log T+\sum_{p<q<X^{3/2}}\frac{\Lambda_X(p^2)}{p\log^2 p}\frac{\Lambda_X(q^2)}{q\log^2 q}\left(T\log\frac{q}{p}+\frac{\log T}{\log\frac{q}{p}}\right)$$

$$\ll T\log T.$$

We shall next evaluate the following sum for $1\le\xi\le X^2$.

$$\begin{aligned}\tilde{S}_{13} &\equiv \sum_{0<\gamma\le T_1}\left(\sigma_{X,(T-\gamma)}-\frac{1}{2}\right)^4 \xi^{(\sigma_{X,(T-\gamma)}-1/2)}\\ &= \sum_{\substack{0<\gamma\le T_1\\ \sigma_{X,(T-\gamma)}-1/2>4/\log X}}\left(\sigma_{X,(T-\gamma)}-\frac{1}{2}\right)^4 \xi^{(\sigma_{X,(T-\gamma)}-1/2)}+O\left(\frac{T\log T}{\log^4 X}\right)\\ &= S_{17}+O\left(\frac{T\log T}{\log^4 X}\right), \quad \text{say}.\end{aligned}$$

As in the estimate of S_{12} in the section 3, we get by our choice of X,

$$S_{17} \ll \sum_{0<\gamma\le T_1}\sum_{\substack{|T-\gamma-\gamma'|<X^{3|\beta'-1/2|}/\log X\\ \beta'-1/2>2/\log X}}\left(\beta'-\frac{1}{2}\right)^4 \xi^{2(\beta'-1/2)}$$

$$\ll \sum_{\substack{0<\gamma'\ll T\\ \beta'-1/2>2/\log X}}\left(\beta'-\frac{1}{2}\right)^4 \xi^{2(\beta'-1/2)}\sum_{\substack{0<\gamma\le T_1\\ |T-\gamma-\gamma'|<X^{3|\beta'-1/2|}/\log X}}\cdot 1$$

$$\ll T.$$

Hence, we get

$$\tilde{S}_{13} \ll \frac{T\log T}{\log^4 X}.$$

Using this and the result in the section 3, we get

$$\sqrt{S_{13}}\sqrt{S_{14}} \ll \sqrt{\frac{T\log T}{\log X}} \cdot \left(\int_{1/2}^{\infty} X^{1/2-\sigma} \sum_{0<\gamma\le T_1} \left| \sum_{p<X^3} \frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+i(T-\gamma)}\log^2 X} \right|^2 d\sigma \right)^{1/2}$$

$$\ll \frac{T\log T}{\log\log T}.$$

Similarly, we get

$$\sqrt{S_{11}}\sqrt{S_{12}} \ll \sqrt{\frac{T\log T}{\log X}} \cdot \left(\int_{1/2}^{\infty} X^{1/2-\sigma} \sum_{0<\gamma\le T_1} \left| \sum_{p<X^3} \frac{\Lambda_X(p)\log(Xp)}{p^{\sigma+i(T-\gamma)}\log p \cdot \log X} \right|^2 d\sigma \right)^{1/2}$$

$$\ll \frac{T\log T\sqrt{\log\log\log T}}{\log\log T}.$$

We find, as a result, the largest contribution to our estimate is

$$S_5 \cdot \log T \ll T\left(\frac{\log T}{\log\log T}\right)^2.$$

Consequently, we get

$$S_1 \ll T\left(\frac{\log T}{\log\log T}\right)^2$$

and

$$\int_0^T R_2(t)dt \ll T\left(\frac{\log T}{\log\log T}\right)^2.$$

This proves (i) of Theorem 3.

§6. Proof of Theorem 4

In this section, we shall give a proof of Theorem 4. However, we shall omit some of the details of the argument. We shall also ignore the dependence on X. We put $\delta = \frac{1}{\log X}$ and evaluate the following integral.

$$S \equiv \sum_{0<\gamma<T} \int_{1+\delta+i}^{1+\delta+i(T-\gamma)} \frac{\zeta'}{\zeta}(s)x^{s+\rho}ds.$$

When $T-\gamma=\gamma'$, we choose T' in $T-\gamma \le T' \le T-\gamma+1$ such that

$$\zeta(s) \ne 0 \qquad \text{in} \quad |\Im(s) - T'| \le \frac{C}{\log T}.$$

Then, we get

$$S=\sum_{0<\gamma<T}\left(\int_{-\infty+iT'}^{1+\delta+iT'}-\int_{-\infty+i}^{1+\delta+i}\right)\frac{\zeta'}{\zeta}(s)x^{s+\rho}ds+2\pi i\sum_{\substack{0<\gamma<T\\ \gamma+1<\gamma+\gamma'<T}}X^{\rho+\rho'}+O(T\log^2 T)$$

$$=2\pi i\sum_{\substack{0<\gamma<T\\ \gamma+1<\gamma+\gamma'<T}}X^{\rho+\rho'}+O(T\log^2 T)\,.$$

On the other hand,

$$S=-iX^{1+\delta}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{1+\delta}}\sum_{0<\gamma<T}X^{\rho}\int_1^{T-\gamma}\frac{1}{n^{it}}X^{it}dt$$

$$=-\Lambda(X)\sum_{0<\gamma<T}X^{\rho}(T-\gamma-1)-iX^{1+\delta}\sum_{\substack{n=2\\ n\neq X}}^{\infty}\frac{\Lambda(n)}{n^{1+\delta}}\sum_{0<\gamma<T}X^{\rho}\frac{\left(\frac{X}{n}\right)^{i(T-\gamma)}-\left(\frac{X}{n}\right)^{i}}{i\log\frac{x}{n}}$$

$$=-i(T-1)\Lambda(X)\sum_{0<\gamma<T}X^{\rho}+i\Lambda(X)\sum_{0<\gamma<T}X^{\rho}\gamma+O(T\log T)$$

$$=\frac{iT^2}{4\pi}\Lambda(X)^2+O(T\log T)\,,$$

since

$$\sum_{0<\gamma<T}X^{\rho}\gamma=-\frac{1}{4\pi}\Lambda(X)T^2+O(T\log T)\,.$$

Combining this with the above evaluation of S, we get

$$\sum_{\substack{0<\gamma,\gamma'<T\\ \gamma+1<\gamma+\gamma'<T}}X^{\rho+\rho'}=\frac{1}{8\pi^2}T^2\Lambda(X)^2+O(T\log^2 T)\,.$$

This proves our Theorem 4.

§7. Proof of Theorem 5

In this section, we assume the Riemann Hypothesis.
Here we shall evaluate the following sum for $X>1$ and for $T>T_0$.

$$S\equiv\sum_{\substack{0<\gamma,\gamma'\leq T\\ \gamma+\gamma'\leq T}}X^{i(\gamma+\gamma')}\,.$$

Using p. 293 of Fujii [14], we get

$$S=\sum_{C<\gamma\leq T-C} X^{i\gamma}\sum_{C<\gamma'\leq T-\gamma} X^{i\gamma'}$$
$$=\sum_{C<\gamma\leq T-C} X^{i\gamma}\left\{M(X,\,T-\gamma)-i\log X\int_1^{T-\gamma}\cos(t\log X)S(t)dt\right.$$
$$\left.+\log X\int_1^{T-\gamma}\sin(t\log X)S(t)dt+X^{i(T-\gamma)}S(T-\gamma)+O(1)\right\}.$$

We put $\delta=\dfrac{1}{\log X}$ and evaluate the integrals in the same manner as in pp. 294–296 of Fujii [14].

$$\int_1^{T-\gamma}\cos(t\log X)S(t)dt$$
$$\equiv\Im\left\{\frac{1}{\pi i}\left(\int_{1+\delta+i}^{1+\delta+i(T-\gamma)}-\int_{1/2+\delta+i(T-\gamma)}^{1+\delta+i(T-\gamma)}+\int_{1/2+i}^{1+\delta+i}\right)\right.$$
$$\left.\cdot\cos\left(-i\left(z-\frac{1}{2}\right)\log X\right)\log\zeta(z)dz\right\}$$
$$=\Im\left\{\frac{1}{\pi i}\frac{i}{2}\frac{\Lambda(X)}{\sqrt{X}\log X}(T-\gamma-1)\right\}+O(\sqrt{X}\log\log(3X))$$
$$+O\left(\sqrt{X}\frac{\log(T-\gamma)}{(\log\log(T-\gamma))^2}\right)+O\left(\frac{\Lambda(P(X))}{\sqrt{X}\log X}\min\left(T-\gamma,\frac{X}{|X-P(X)|}\right)\right),$$

where $P(X)$ is defined in the statement of Lemma 2 in the introduction.

Thus we get

$$\log X\cdot\int_1^{T-\gamma}\cos(t\log X)S(t)dt\ll\sqrt{X}\frac{\log X\cdot\log(T-\gamma)}{(\log\log(T-\gamma))^2}+\sqrt{X}\log X\cdot\log\log(3X)$$
$$+\frac{\Lambda(P(X))}{\sqrt{X}}\min\left(T-\gamma,\frac{X}{|X-P(X)|}\right).$$

In the same manner, we get

$$\begin{aligned}
&\log X\cdot\int_1^{T-\gamma}\sin(t\log X)S(t)dt\\
&=\log X\cdot\mathfrak{I}\left\{\frac{1}{\pi i}\frac{1}{2}\frac{\Lambda(X)}{\sqrt{X}\log X}(T-\gamma-1)\right\}\\
&\quad+O\left(\sqrt{X}\frac{\log X\cdot\log(T-\gamma)}{(\log\log(T-\gamma))^2}\right)+O(\sqrt{X}\log X\log\log(3X))\\
&\quad+O\left(\frac{\Lambda(P(X))}{\sqrt{X}}\min\left(T-\gamma,\frac{X}{|X-P(X)|}\right)\right)\\
&=-\frac{1}{2\pi}\frac{\Lambda(X)}{\sqrt{X}}(T-\gamma-1)\\
&\quad+O\left(\sqrt{X}\frac{\log(T-\gamma)}{(\log\log(T-\gamma))^2}\right)+O(\sqrt{X}\log X\log\log(3X))\\
&\quad+O\left(\frac{\Lambda(P(X))}{\sqrt{X}}\min\left(T-\gamma,\frac{X}{|X-P(X)|}\right)\right).
\end{aligned}$$

Hence, we get

$$\begin{aligned}
S=&\sum_{C<\gamma\le T-C}X^{i\gamma}M(X,T-\gamma)+X^{iT}\sum_{C<\gamma\le T-C}S(T-\gamma)\\
&-\frac{1}{2\pi}\frac{\Lambda(X)}{\sqrt{X}}\sum_{C<\gamma\le T-C}X^{i\gamma}(T-\gamma-1)\\
&+O\left(T\log T\left(\sqrt{X}\log X\cdot\log\log(3X)+\frac{\Lambda(P(X))}{\sqrt{X}}\min\left(T,\frac{X}{|X-P(X)|}\right)\right)\right)\\
&+O\left(\sqrt{X}\log X\sum_{C<\gamma\le T-C}\frac{\log(T-\gamma)}{(\log\log(T-\gamma))^2}\right)\\
=&S_1+S_2+S_3\\
&+O\left(T\log T\left(\sqrt{X}\log X\cdot\log\log(3X)+\frac{\Lambda(P(X))}{\sqrt{X}}\min\left(T,\frac{X}{|X-P(X)|}\right)\right)\right)\\
&+O\left(\sqrt{X}\log X\cdot T\left(\frac{\log T}{\log\log T}\right)^2\right),\quad\text{say}.
\end{aligned}$$

By the section 3, we get

$$S_2\ll T\log T.$$

By the definition, we get

$$S_1=\frac{1}{2\pi}\sum_{C<\gamma\le T-C}X^{i\gamma}\int_1^{T-\gamma}X^{it}\log\frac{t}{2\pi}\,dt$$
$$=\frac{1}{2\pi}\int_1^{T-C}\Bigg(\sum_{\substack{C<\gamma\le T-t\\ \gamma\le T-C}}X^{i\gamma}\Bigg)X^{it}\log\frac{t}{2\pi}\,dt$$
$$=\frac{1}{2\pi}\int_C^{T-C}\Bigg(\sum_{C<\gamma\le T-t}X^{i\gamma}\Bigg)X^{it}\log\frac{t}{2\pi}\,dt+O(T\log T)\,.$$

Applying Lemma 2 to the last inner sum, we get simply

$$S_1=\frac{1}{2\pi}\int_C^{T-C}\left(-\frac{T-t}{2\pi}\right)\frac{\Lambda(X)}{\sqrt{X}}X^{it}\log\frac{t}{2\pi}\,dt$$
$$+\frac{1}{2\pi}\int_C^{T-C}M(X,\,T-t)X^{it}\log\frac{t}{2\pi}\,dt$$
$$+\frac{1}{2\pi}\int_C^{T-C}X^{i(T-t)}S(T-t)X^{it}\log\frac{t}{2\pi}\,dt$$
$$+O\left(T\log T\left(\frac{\log(3X)}{\sqrt{X}}\min\left(T,\frac{X}{|X-P(X)|}\right)+\sqrt{X}\log(3X)\cdot\log\log(3X)\right)\right)$$
$$+O\left(T\left(\frac{\log T}{\log\log T}\right)^2\sqrt{X}\log(3X)\right).$$

We denote the last three terms involving the integral by S_4, S_5 and S_6, respectively. By simple calculations, we can evaluate these as follows.

$$S_4=\frac{\Lambda(X)}{\sqrt{X}}\frac{1}{2\pi}\left\{\left[\frac{X^{it}}{i\log X}\left(-\frac{T-t}{2\pi}\right)\log\frac{t}{2\pi}\right]_C^{T-C}\right.$$
$$\left.-\int_C^{T-C}\frac{X^{it}}{i\log X}\left(-\frac{T-t}{2\pi}\right)\frac{1}{t}\,dt-\int_C^{T-C}\frac{X^{it}}{i\log X}\left(\frac{1}{2\pi}\right)\log\frac{t}{2\pi}\,dt\right\}$$
$$\ll\frac{\Lambda(X)}{\sqrt{X}\log X}T\log T\,.$$

$$S_6=X^{iT}\frac{1}{2\pi}\int_C^{T-C}S(T-t)\log\frac{t}{2\pi}\,dt=X^{iT}\frac{1}{2\pi}\int_C^{T-C}S(t)\log\frac{T-t}{2\pi}\,dt$$
$$=X^{iT}\frac{1}{2\pi}\left\{\left[S_1(t)\log\frac{T-t}{2\pi}\right]_C^{T-C}+\int_C^{T-C}S_1(t)\frac{1}{T-t}\,dt\right\}$$
$$=O(\log^2 T)\,.$$

$$S_5=\frac{1}{2\pi}\int_C^{T-C}\left(\frac{1}{2\pi}\int_1^{T-t}X^{it}\log\frac{t}{2\pi}\,dt\right)X^{it}\log\frac{t}{2\pi}\,dt$$
$$=\left[\frac{1}{4\pi^2}\frac{X^{it}}{i\log X}\log\frac{t}{2\pi}\left(\int_1^{T-t}X^{it}\log\frac{t}{2\pi}\,dt\right)\right]_C^{T-C}$$
$$-\frac{1}{4\pi^2}\int_C^{T-C}\frac{X^{it}}{i\log X}\frac{1}{t}\left(\int_1^{T-t}X^{it}\log\frac{t}{2\pi}\,dt\right)dt$$
$$+\frac{1}{4\pi^2}\int_C^{T-C}\frac{X^{it}}{i\log X}\log\frac{t}{2\pi}X^{i(T-t)}\log\frac{T-t}{2\pi}\,dt$$
$$=O\left(\left(\frac{\log T}{\log X}\right)^2\right)+\frac{X^{iT}}{4\pi^2 i\log X}\int_C^{T-C}\log\frac{t}{2\pi}\log\frac{T-t}{2\pi}\,dt\,.$$

The last integral is

$$=\left[t\log\frac{t}{2\pi}\log\frac{T-t}{2\pi}\right]_C^{T-C}-\int_C^{T-C}\log\frac{T-t}{2\pi}\,dt+\int_C^{T-C}\frac{t}{T-t}\log\frac{t}{2\pi}\,dt$$
$$=O(T\log T)+\frac{1}{T}\sum_{n=0}^{\infty}\frac{1}{T^n}\int_C^{T-C}t^{n+1}\log\frac{t}{2\pi}\,dt$$
$$=O(T\log T)+\frac{1}{T}\sum_{n=0}^{\infty}\frac{1}{T^n}\frac{(T-C)^{n+2}}{n+2}\log\frac{T-C}{2\pi}$$
$$-\frac{1}{T}\sum_{n=0}^{\infty}\frac{1}{T^n}\frac{(T-C)^{n+2}}{(n+2)^2}+\frac{1}{T}\sum_{n=0}^{\infty}\frac{1}{T^n}O\left(\frac{C^{n+2}}{n+2}\right)$$
$$=O(T\log T)+T\log\frac{T-C}{2\pi}\left(\sum_{n=0}^{\infty}\left(1-\frac{C}{T}\right)^n\frac{1}{n}-\left(1-\frac{C}{T}\right)-\frac{1}{2}\left(1-\frac{C}{T}\right)^2\right)$$
$$=T\log^2 T+O(T\log T)\,.$$

Hence, we get

$$S_5=\frac{X^{iT}}{4\pi^2 i\log X}T\log^2 T+O\left(\frac{T\log T}{\log X}\right)+O\left(\left(\frac{\log T}{\log X}\right)^2\right).$$

Consequently, we get

$$S_1=\frac{X^{iT}}{4\pi^2 i\log X}T\log^2 T+O\left(\frac{T\log T}{\log X}\right)+O\left(\left(\frac{\log T}{\log X}\right)^2\right)$$
$$+O\left(T\log T\left(\frac{\log(3X)}{\sqrt{X}}\min\left(T,\frac{X}{|X-P(X)|}\right)+\sqrt{X}\log(3X)\log\log(3X)\right)\right)$$
$$+O\left(T\left(\frac{\log T}{\log\log T}\right)^2\sqrt{X}\log(3X)\right).$$

Finally, we shall evaluate S_3.

$$S_3 = -\frac{1}{2\pi}\frac{\Lambda(X)}{\sqrt{X}}\int_C^{T-C}\sum_{C<\gamma\le t} X^{i\gamma}dt + O\left(\frac{\Lambda(X)}{\sqrt{X}}T\log T\right)$$

$$= S_7 + O\left(\frac{\Lambda(X)}{\sqrt{X}}T\log T\right), \quad \text{say}.$$

Applying directly Lemma 2, we get

$$S_7 = -\frac{1}{2\pi}\frac{\Lambda(X)}{\sqrt{X}}\int_C^T\left\{-\frac{t}{2\pi}\frac{\Lambda(X)}{\sqrt{X}} + M(X, t) + X^{it}S(t)\right.$$

$$\left. + O(\sqrt{X}\log(3X)\log\log(3X)) + O\left(\sqrt{X}\log X\cdot\frac{\log T}{(\log\log T)^2}\right)\right\}dt$$

$$= \frac{1}{8\pi^2}\frac{\Lambda(X)^2}{X}T^2 + O\left(\frac{\Lambda(X)}{\sqrt{X}}\left\{T\log T + T\sqrt{X}\log(3X)\log\log(3X)\right.\right.$$

$$\left.\left. + \sqrt{X}\log X\cdot\frac{T\log T}{(\log\log T)^2}\right\}\right).$$

Combining all of our evaluations, we get

$$S = \frac{1}{8\pi^2}\frac{\Lambda(X)^2}{X}T^2 + \frac{X^{iT}}{4\pi^2 i\log X}T\log^2 T$$

$$+ O\left(T\left(\frac{\log T}{\log\log T}\right)^2\sqrt{X}\log(3X)\right)$$

$$+ O\left(T\log T\left(\sqrt{X}\log(3X)\log\log(3X) + \frac{1}{\log X}\right.\right.$$

$$\left.\left. + \frac{\log(3X)}{\sqrt{X}}\min\left(T, \frac{X}{|X-P(X)|}\right)\right)\right) + O\left(\left(\frac{\log T}{\log X}\right)^2\right).$$

This proves our Theorem 5.

§8. Proof of Theorems 6, 7 and 8—part (I)

We suppose from the section 8 till the section 12 that $0<\alpha\ll T$, $0<b\le 2$ and $T_0<T\le Y\le 2T$.

We put $f(y) = by\log\frac{y}{2\pi e\alpha}$ and $\delta = \frac{1}{\log T}$. We consider the sum

$$S \equiv \sum_{0<\gamma<Y-T}\int_{1+\delta+i(\gamma+1)}^{1+\delta+i(\gamma+T)}\frac{\zeta'}{\zeta}(s-i\gamma)e^{if(-i(s-1/2))}ds.$$

We choose T' in $T \leq T' \leq T+1$ such that

$$\frac{\zeta'}{\zeta}(\sigma+i(T'+\gamma)-i\gamma) \ll \log^2 T \qquad \text{in} \quad -1 \leq \sigma \leq 2 .$$

$$S = \sum_{0<\gamma<Y-T}\left(-\int_{1+\delta+i(T'+\gamma)}^{-\delta+i(T'+\gamma)} + \int_{-\delta+i(\gamma+1)}^{-\delta+i(T'+\gamma)} - \int_{-\delta+i(\gamma+1)}^{1+\delta+i(\gamma+1)}\right)$$

$$\cdot \frac{\zeta'}{\zeta}(s-i\gamma)e^{if(-i(s-1/2))}ds$$

$$+2\pi i \sum_{\substack{0<\gamma<Y-T \\ 0<\gamma'<T'}} e^{if(\gamma+\gamma')}$$

$$+O\left(\sum_{0<\gamma<Y-T}\int_{1+\delta+i(T+\gamma)}^{1+\delta+i(T'+\gamma)}\left|\frac{\zeta'}{\zeta}(s-i\gamma)\right||e^{if(-i(s-1/2))}||ds|\right)$$

$$=(S_1+S_2+S_3)+S_4+O(S_5), \quad \text{say} .$$

We get simply,

$$S_5 \ll T^{1+b/2}\log^2 T \cdot \alpha^{-b/2} .$$

It is obvious to have

$$S_4 = 2\pi i \sum_{\substack{0<\gamma<Y-T \\ 0<\gamma'\leq T}} e^{if(\gamma+\gamma')} + O(T\log^2 T) .$$

Applying the above estimate on $\frac{\zeta'}{\zeta}(s)$, we get

$$S_1 \ll \log^2 T \sum_{0<\gamma<Y-T}\int_{1+\delta}^{-\delta} | e^{if(-i(\sigma-1/2+i(T'+\gamma)))}|d\sigma$$

$$\ll T^{1+b/2}\log^3 T \cdot \alpha^{-b/2} + T^{1-b/2}\log^3 T \cdot \alpha^{b/2} .$$

Similarly, we get

$$S_3 \ll T^{1+b/2}\log^3 T \cdot \alpha^{-b/2} + T^{1-b/2}\log^3 T \cdot \alpha^{b/2} .$$

By the functional equation, we get

$$S_2 = \sum_{0<\gamma<Y-T}\int_{-\delta+i(\gamma+1)}^{-\delta+i(T'+\gamma)}\left(\log(2\pi)+\frac{\pi}{2}\tan\left(\frac{1}{2}(s-i\gamma)\pi\right)\right.$$

$$\left.-\frac{\Gamma'}{\Gamma}(s-i\gamma)-\frac{\zeta'}{\zeta}(1-s+i\gamma)\right)e^{if(-i(s-1/2))}ds$$

$$=S_6+S_7+S_8+S_9, \quad \text{say} .$$

We remark that for $s=-\delta+it$

$$\log(2\pi)+\frac{\pi}{2}\tan\left(\frac{1}{2}(s-i\gamma)\pi\right)-\frac{\Gamma'}{\Gamma}(s-i\gamma)=\log(2\pi)-\log(t-\gamma)+O\left(\frac{1}{t-\gamma}\right)$$

and

$$e^{if(-i(s-1/2))}=\left(\frac{t}{2\pi\alpha}\right)^{-b(1/2+\delta)}e^{ibt\log(t/2\pi e\alpha)}+O(t^{-1-b/2-\delta b}\alpha^{b(1/2+\delta)}).$$

Hence we get

$$\begin{aligned}S_6+S_7+S_8&=i\sum_{0<\gamma<Y-T}\int_{\gamma+1}^{T'+\gamma}\left(\log(2\pi)-\log(t-\gamma)+O\left(\frac{1}{t-\gamma}\right)\right)\\&\quad\cdot\left(\left(\frac{t}{2\pi\alpha}\right)^{-b(1/2+\delta)}e^{ibt\log(t/2\pi e\alpha)}+O(t^{-1-b/2-\delta b}\alpha^{b(1/2+\delta)})\right)dt\\&=i\log(2\pi)\sum_{0<\gamma<Y-T}\int_{\gamma+1}^{T'+\gamma}\left(\frac{t}{2\pi\alpha}\right)^{-b(1/2+\delta)}e^{ibt\log(t/2\pi e\alpha)}dt\\&\quad-i\sum_{0<\gamma<Y-T}\int_{\gamma+1}^{T'+\gamma}\log(t-\gamma)\left(\frac{t}{2\pi\alpha}\right)^{-b(1/2+\delta)}e^{ibt\log(t/2\pi e\alpha)}dt\\&\quad+O\left(\alpha^{b(1/2+\delta)}\sum_{0<\gamma<Y-T}\int_{\gamma+1}^{T'+\gamma}t^{-1-b(1/2+\delta)}\left(1+\log(t-\gamma)+O\left(\frac{1}{t-\gamma}\right)\right)dt\right)\\&\quad+\sum_{0<\gamma<Y-T}\int_{\gamma+1}^{T'+\gamma}\left(\frac{t}{2\pi\alpha}\right)^{-b(1/2+\delta)}O\left(\frac{1}{t-\gamma}\right)dt\\&=i\log(2\pi)(2\pi\alpha)^{b(1/2+\delta)}S_{10}-i(2\pi\alpha)^{b(1/2+\delta)}S_{11}+O(S_{12})+O(S_{13}),\quad\text{say}.\end{aligned}$$

We get simply,

$$S_{12}+S_{13}\ll\alpha^{b/2}T^{1-b/2}\log^2 T.$$

$$S_{11}=\sum_{0<\gamma<Y-T}\int_{\gamma+1}^{T+\gamma}\log(t-\gamma)t^{-b(1/2+\delta)}e^{ibt\log(t/2\pi e\alpha)}dt+O(T^{1-b/2}\log^2 T).$$

We put $\varepsilon=T^{-2/5}$ and decompose the last sum into

$$\sum_{\substack{0<\gamma<Y-T\\2\pi\alpha(1+\varepsilon)\le\gamma+1}}+\sum_{\substack{0<\gamma<Y-T\\2\pi\alpha(1-\varepsilon)\le\gamma+1\le2\pi\alpha(1+\varepsilon)}}+\sum_{\substack{0<\gamma<Y-T\\\gamma+1<2\pi\alpha(1-\varepsilon)\le2\pi\alpha(1+\varepsilon)<T+\gamma}}$$

$$+\sum_{\substack{0<\gamma<Y-T\\2\pi\alpha(1-\varepsilon)\le T+\gamma<2\pi\alpha(1+\varepsilon)}}+\sum_{\substack{0<\gamma<Y-T\\T+\gamma\le2\pi\alpha(1-\varepsilon)}}=Z_1+Z_2+Z_3+Z_4+Z_5,\quad\text{say},$$

where some sum may be empty. In particular, if α is sufficiently small, then

$$Z_2+Z_3+Z_4+Z_5=0.$$

Since

$$\left(bt\log\frac{t}{2\pi e\alpha}\right)'=b\log\frac{t}{2\pi\alpha}\geq b\log\frac{\gamma+1}{2\pi\alpha}\geq b\log(1+\varepsilon)$$

for Z_1 and

$$b\log\frac{t}{2\pi\alpha}\leq b\log\frac{T+\gamma}{2\pi\alpha}\leq b\log(1-\varepsilon)$$

for Z_5, by Lemma 4.2 of Titchmarsh [35], we get

$$\begin{aligned}Z_1+Z_5&\ll\frac{1}{\varepsilon}\sum_{0<\gamma\ll T}\gamma^{-b/2}\\&\ll\begin{cases}T^{1-b/2+2/5}\log T & \text{if } 0<b<2\\ \log^2 T & \text{if } b=2\end{cases}\\&\ll T^{1-b/2+2/5}\log T.\end{aligned}$$

Hereafter we suppose that $\alpha\gg 1$ when we treat Z_2, Z_3 and Z_4.

$$\begin{aligned}Z_3&=\sum_{\gamma\in Z_3}\left(\int_{\gamma+1}^{2\pi\alpha(1-\varepsilon)}+\int_{2\pi\alpha(1-\varepsilon)}^{2\pi\alpha}+\int_{2\pi\alpha}^{2\pi\alpha(1+\varepsilon)}+\int_{2\pi\alpha(1+\varepsilon)}^{T+\gamma}\right)\\&=\sum_{\gamma\in Z_3}(J_1+J_2+J_3+J_4)\,,\quad\text{say}\,,\end{aligned}$$

where $\gamma\in Z_3$ means that γ satisfies the conditions in Z_3.

$$\begin{aligned}\sum_{\gamma\in Z_3}J_3&=\sum_{\gamma\in Z_3}\int_{2\pi\alpha}^{2\pi\alpha(1+\varepsilon)}\log t\cdot t^{-b(1/2+\delta)}e^{ibt\log(t/2\pi e\alpha)}dt\\&\quad-\sum_{\gamma\in Z_3}\sum_{k=1}^{\infty}\frac{\gamma^k}{k}\int_{2\pi\alpha}^{2\pi\alpha(1+\varepsilon)}t^{-(k+b(1/2+\delta))}e^{ibt\log(t/2\pi e\alpha)}dt\\&=\sum_{\gamma\in Z_3}J_3'+\sum_{\gamma\in Z_3}J_3''\,,\quad\text{say}\,.\end{aligned}$$

We shall treat $\sum_{\gamma\in Z_3}J_3''$ first.

$$\begin{aligned}\int_{2\pi\alpha}^{2\pi\alpha(1+\varepsilon)}t^{-(k+b(1/2+\delta))}e^{ibt\log(t/2\pi e\alpha)}dt&=(2\pi\alpha)^{-(k+b(1/2+\delta))+1}\left(e^{-2\pi i\alpha b}\int_0^{\varepsilon}e^{i\pi\alpha b\mu^2}d\mu\right.\\&\qquad\left.+\int_0^{\varepsilon}(O(\alpha\mu^3)+O(k\mu)+O(\alpha k\mu^4))d\mu\right).\end{aligned}$$

If $\varepsilon\sqrt{\alpha}\gg 1$, then this is

$$=(2\pi\alpha)^{-(k+b(1/2+\delta))+1}e^{-2\pi i\alpha b}\left(\frac{1}{\sqrt{2b\alpha}(1-i)}+O\left(\frac{1}{\varepsilon\alpha}\right)+O(\alpha\varepsilon^4+k\varepsilon^2+\alpha k\varepsilon^5)\right).$$

If $\varepsilon\sqrt{\alpha}\ll 1$, then this is

$$=O((2\pi\alpha)^{-(k+b(1/2+\delta))+1}(\varepsilon+k\varepsilon^2)).$$

Consequently, if $\varepsilon\sqrt{\alpha}\gg 1$, then

$$\sum_{\gamma\in Z_3} J_3''=-\sum_{\gamma\in Z_3}\sum_{k=1}^{\infty}\frac{\gamma^k}{k}\left\{(2\pi\alpha)^{-(k+b(1/2+\delta))+1}e^{-2\pi i\alpha b}\left(\frac{1}{\sqrt{2b\alpha}(1-i)}+O\left(\frac{1}{\varepsilon\alpha}\right)\right.\right.$$

$$\left.\left.+O(\alpha\varepsilon^4+k\varepsilon^2+\alpha k\varepsilon^5)\right)\right\}$$

$$=(2\pi\alpha)^{-b(1/2+\delta)+1}e^{-2\pi i\alpha b}\frac{1}{\sqrt{2b\alpha}(1-i)}\sum_{\gamma\in Z_3}\log\left(1-\frac{\gamma}{2\pi\alpha}\right)$$

$$+O(\alpha^{2-b/2}\log^2\alpha\cdot T^{-2/5}).$$

If $\varepsilon\sqrt{\alpha}\ll 1$, then

$$\sum_{\gamma\in Z_3} J_3''=O\left(\sum_{\gamma\in Z_3}\sum_{k=1}^{\infty}\frac{\gamma^k}{k}(2\pi\alpha)^{-(k+b(1/2+\delta))+1}(\varepsilon+k\varepsilon^2)\right)$$

$$=O(\varepsilon\alpha^{2-b/2}\log\alpha\log T).$$

We shall next treat the sum $\sum_{\gamma\in Z_3} J_3'$.

$$\sum_{\gamma\in Z_3} J_3'=\sum_{\gamma\in Z_3}\int_{2\pi\alpha}^{2\pi\alpha(1+\varepsilon)}\log\frac{t}{2\pi\alpha}\cdot t^{-(b(1/2+\delta))}e^{ibt\log(t/2\pi e\alpha)}dt$$

$$+\log(2\pi\alpha)\sum_{\gamma\in Z_3}\int_{2\pi\alpha}^{2\pi\alpha(1+\varepsilon)}t^{-(b(1/2+\delta))}e^{ibt\log(t/2\pi e\alpha)}dt$$

$$=\sum_{\gamma\in Z_3} J_3'(I)+\sum_{\gamma\in Z_3} J_3'(II),\quad \text{say}.$$

$$\sum_{\gamma\in Z_3} J_3'(I)=\frac{1}{ib}\sum_{\gamma\in Z_3}\int_{2\pi\alpha}^{2\pi\alpha(1+\varepsilon)}t^{-(b(1/2+\delta))}(e^{ibt\log(t/2\pi e\alpha)})'dt\ll\alpha^{1-b/2}\log\alpha.$$

The integral in $\sum_{\gamma\in Z_3} J_3'(II)$ can be treated as above and we get if $\varepsilon\sqrt{\alpha}\gg 1$, then

$$\sum_{\gamma\in Z_3} J_3'(II)=\left(\sum_{\gamma\in Z_3}\cdot 1\right)\log(2\pi\alpha)\left\{(2\pi\alpha)^{-b(1/2+\delta))+1}e^{-2\pi i\alpha b}\frac{1}{\sqrt{2b\alpha}(1-i)}\right.$$

$$\left.+O(T^{2/5}\alpha^{-b/2})\right\}.$$

If $\varepsilon\sqrt{\alpha}\ll 1$, then

$$\sum_{\gamma\in Z_3} J_3'(II)=O\left(\left(\sum_{\gamma\in Z_3}\cdot 1\right)\varepsilon\alpha^{-(b(1/2+\delta))+1}\log(2\pi\alpha)\ll\varepsilon\alpha^{2-b/2}\log^2\alpha.\right.$$

We can treat the sum $\sum_{\gamma\in Z_3} J_2$ in the same manner.

$$\sum_{\gamma\in Z_3} J_4 \ll \frac{1}{\varepsilon} \sum_{\gamma\le 2\pi\alpha(1-\varepsilon)} \alpha^{-b(1/2+\delta)} \log(2\pi\alpha(1+\varepsilon)-\gamma) \ll \frac{1}{\varepsilon}\alpha^{1-b/2}\log\alpha\log T\,.$$

Similarly, we get

$$\sum_{\gamma\in Z_3} J_1 \ll \frac{1}{\varepsilon}\alpha^{1-b/2}\log^2\alpha\,.$$

$$Z_2 \ll \sum_{\gamma\in Z_2}\left(\int_{\gamma+1}^{2\pi\alpha(1+\varepsilon)} + \int_{2\pi\alpha(1+\varepsilon)}^{T+\gamma}\right) \ll \alpha^{-b/2}\log\alpha\cdot\left(\varepsilon\alpha+\frac{1}{\varepsilon}\right)(\varepsilon\alpha\log\alpha+\log\alpha)\,.$$

$$Z_4 \ll \sum_{\gamma\in Z_4}\left(\int_{\gamma+1}^{2\pi\alpha(1-\varepsilon)} + \int_{2\pi\alpha(1-\varepsilon)}^{T+\gamma}\right) \ll T^{1+2/5-b/2}\log T\,.$$

Combining the evaluations of Z_1, Z_2, Z_3, Z_4 and Z_5, if $T^{4/5} \ll \alpha \ll T$, then we get

$$\begin{aligned}
-i(2\pi\alpha)^{b(1/2+\delta)}S_{11} &= -2i\frac{2\pi\alpha\cdot e^{-2\pi i\alpha b}}{\sqrt{2b\alpha}(1-i)}\sum_{0<\gamma<\min(Y-T,2\pi\alpha(1-\varepsilon))}\log\left(1-\frac{\gamma}{2\pi\alpha}\right)\\
&\quad -2i\frac{2\pi\alpha\cdot e^{-2\pi i\alpha b}\log(2\pi\alpha)}{\sqrt{2b\alpha}(1-i)}\sum_{0<\gamma<\min(Y-T,2\pi\alpha)}\cdot 1\\
&\quad + O(\alpha^{b/2}T^{1+2/5-b/2}\log T + T^{2/5}\alpha\log\alpha\cdot\log T + T^{-2/5}\alpha^2\log^2\alpha)\\
&= O(\alpha^{3/2}\log^2\alpha) + O(\alpha^{b/2}T^{1+2/5-b/2}\log T\\
&\quad + T^{2/5}\alpha\log\alpha\cdot\log T + T^{-2/5}\alpha^2\log^2\alpha)\,.
\end{aligned}$$

If $0<\alpha\ll T^{4/5}$, then we get

$$-i(2\pi\alpha)^{b(1/2+\delta)}S_{11} = O(\alpha^{b/2}T^{1+2/5-b/2}\log T + T^{2/5}(\alpha+3)\log(\alpha+3)\cdot\log T)\,.$$

In a similar manner, we can evaluate the sum S_{10}.

If $T^{4/5}\ll\alpha\ll T$, then we get

$$\begin{aligned}
i\log(2\pi)(2\pi\alpha)^{b(1/2+\delta)}S_{10} &= 2i\log(2\pi)\frac{2\pi\alpha\cdot e^{-2\pi i\alpha b}}{\sqrt{2b\alpha}(1-i)}\sum_{0<\gamma<\min(Y-T,2\pi\alpha)}\cdot 1\\
&\quad + O(\alpha^{b/2}T^{1+2/5-b/2}\log T + T^{2/5}\alpha\log\alpha\cdot\log T + T^{-2/5}\alpha^2\log^2\alpha)\\
&= O(\alpha^{3/2}\log\alpha) + O(\alpha^{b/2}T^{1+2/5-b/2}\log T + T^{2/5}\alpha\log\alpha\cdot\log T + T^{-2/5}\alpha^2\log^2\alpha)\,.
\end{aligned}$$

If $0<\alpha\ll T^{4/5}$, then we get

$$i\log(2\pi)(2\pi\alpha)^{b(1/2+\delta)}S_{10} = O(\alpha^{b/2}T^{1+2/5-b/2}\log T + T^{2/5}(\alpha+3)\log(\alpha+3)\cdot\log T)\,.$$

§9. Proof of Theorems 6, 7 and 8—part (II)

In this section we shall evaluate S_9. We put $B=\dfrac{1}{b}$.

$$S_9 = i \sum_{0<\gamma<Y-T} \sum_{k=2}^{\infty} \frac{\Lambda(k)}{k^{1+\delta+i\gamma}} \int_{\gamma}^{T+\gamma} \left(\frac{t}{2\pi\alpha}\right)^{-b(1/2+\delta)} e^{ibt\log(tk^B/2\pi e\alpha)}\,dt$$
$$+ O\left(\alpha^{b/2} \sum_{0<\gamma<Y-T} \sum_{k=2}^{\infty} \frac{\Lambda(k)}{k^{1+\delta+i\gamma}} \int_{\gamma}^{T+\gamma} t^{-1-b/2-\delta b}\,dt\right)$$
$$+ O(\alpha^{b/2} T^{1-b/2} \log^2 T)$$
$$= S_{14} + O(S_{15}) + O(\alpha^{b/2} T^{1-b/2} \log^2 T)\,, \quad \text{say}\,.$$

We get trivially,

$$S_{15} \ll \alpha^{b/2} T^{1-b/2} \log^2 T\,.$$

We shall treat S_{14} by decomposing it into two parts as follows.

$$S_{14} = \left(\sum_{\substack{0<\gamma<Y-T\\ 2\pi\alpha\le\gamma}} + \sum_{\substack{0<\gamma<Y-T\\ 2\pi\alpha>\gamma}}\right) \sum_{k=2}^{\infty} \frac{\Lambda(k)}{k^{1+\delta+i\gamma}} \int_{\gamma}^{T+\gamma} \left(\frac{t}{2\pi\alpha}\right)^{-b(1/2+\delta)} e^{ibt\log(tk^B/2\pi e\alpha)}\,dt$$
$$= S_{16} + S_{17}\,, \quad \text{say}\,.$$

Since for S_{16} and for $\gamma \le t \le T+\gamma$,

$$\frac{d}{dt}\left(bt\log\frac{tk^B}{2\pi e\alpha}\right) = b\log\frac{tk^B}{2\pi\alpha} \ge b\log\frac{\gamma k^B}{2\pi\alpha} \ge \log k\,,$$

we get by Lemma 4.2 of Titchmarsh [35]

$$S_{16} \ll \alpha^{b(1/2+\delta)} \sum_{\gamma\ll T} \gamma^{-b/2} \log\log T$$
$$\ll \begin{cases} \alpha^{b/2} T^{1-b/2} \log T \cdot \log\log T & \text{if } \ 0<b<2 \\ \alpha^{b/2} \log^2 T \cdot \log\log T & \text{if } \ b=2\,. \end{cases}$$

When α is sufficiently small, $S_{17}=0$. Hence, we may suppose that $\alpha \gg 1$.

To treat S_{17}, we put $v_1 = \left(\frac{2\pi\alpha}{T+\gamma}\right)^b$ and $v = \left(\frac{2\pi\alpha}{\gamma}\right)^b$.

$$S_{17} = \sum_{\gamma\in S_{17}} \left(\sum_{k\le v_1-1} + \sum_{v_1-1<k\le v_1} + \sum_{v_1<k<v} + \sum_{v\le k<v+1} + \sum_{k\ge v+1}\right)$$
$$= \sum_{\gamma\in S_{17}} (A_1+A_2+A_3+A_4+A_5)\,, \quad \text{say}\,,$$

where some sum may be empty and $\gamma \in S_{17}$ means, by our convention, that $0<\gamma<Y-T$ and $2\pi\alpha>\gamma$. Since $\alpha \ll T$, we may take $v_1=1$ and

$$\sum_{\gamma\in S_{17}} \sum_{k\in A_1\cup A_2} = 0\,.$$

Since for $k \in A_5$ and for $\gamma \le t \le T+\gamma$

$$\frac{d}{dt}\left(bt\log\frac{tk^B}{2\pi e\alpha}\right)=b\log\frac{tk^B}{2\pi\alpha}\geq b\log\frac{\gamma k^B}{2\pi\alpha}=\log\frac{k}{v}\geq\log\frac{v+1}{v}>0\,,$$

we get

$$\sum_{\gamma\in S_{17}}\sum_{k\in A_5}\frac{\Lambda(k)}{k^{1+\delta+i\gamma}}\int_{\gamma}^{T+\gamma}\left(\frac{t}{2\pi\alpha}\right)^{-b(1/2+\delta)}e^{ibt\log(tk^B/2\pi e\alpha)}dt$$

$$\ll\alpha^{b/2}\sum_{\gamma\ll\alpha}\gamma^{-b/2}\sum_{k\geq v+1}\frac{\Lambda(k)}{k^{1+\delta+i\gamma}|\log\frac{v}{k}|}$$

$$\ll\alpha^{b/2}\sum_{\gamma\ll\alpha}\gamma^{-b/2}\sum_{k>2v}\frac{\Lambda(k)}{k^{1+\delta}}+\alpha^{b/2}\sum_{\gamma\ll\alpha}\gamma^{-b/2}\sum_{v+1\leq k\leq 2v}\frac{\Lambda(k)}{k^{1+\delta}}\frac{k}{|k-v|}$$

$$\ll\alpha^{b/2}\log T\cdot\log\log T\cdot\sum_{\gamma\ll\alpha}\gamma^{-b/2}$$

$$\ll\alpha^{b/2}\log T\cdot\log\log T\cdot\begin{cases}\alpha^{1-b/2}\log\alpha & \text{if}\quad 0<b<2\\ \log^2\alpha & \text{if}\quad b=2\end{cases}$$

$$\ll T\log^3 T\cdot\log\log T\,.$$

We put $\varepsilon=T^{-2/5}$ and $\xi_k=\frac{2\pi\alpha}{k^B}$ and decompose A_3 further as follows.

$$A_3=\sum_{\substack{v_1<k<v\\ \xi_k(1-\varepsilon)\leq\gamma}}+\sum_{\substack{v_1<k<v\\ \gamma<\xi_k(1-\varepsilon)<\xi_k(1+\varepsilon)<T+\gamma}}+\sum_{\substack{v_1<k<v\\ T+\gamma\leq\xi_k(1+\varepsilon)}}$$

$$=A_3'+A_3''+A_3'''\,,\quad\text{say}\,.$$

We shall first treat A_3''.

$$J(k)\equiv(2\pi\alpha)^{b(1/2+\delta)}\int_{\gamma}^{T+\gamma}t^{-b(1/2+\delta)}e^{ibt\log(tk^B/2\pi e\alpha)}dt$$

$$=(2\pi\alpha)^{b(1/2+\delta)}\left(\int_{\gamma}^{\xi_k(1-\varepsilon)}+\int_{\xi_k(1-\varepsilon)}^{\xi_k}+\int_{\xi_k}^{\xi_k(1+\varepsilon)}+\int_{\xi_k(1+\varepsilon)}^{T+\gamma}\right)$$

$$\cdot\,t^{-b(1/2+\delta)}e^{ibt\log(tk^B/2\pi e\alpha)}dt$$

$$=J_1(k)+J_2(k)+J_3(k)+J_4(k)\,,\quad\text{say}\,.$$

$$\sum_{\gamma\in S_{17}}\sum_{k\in A_3''}\frac{\Lambda(k)}{k^{1+\delta+i\gamma}}J_1(k)\ll\alpha^{b/2}\frac{1}{\varepsilon}\log T\sum_{\gamma\in S_{17}}\gamma^{-b/2}\ll\alpha T^{2/5}\log^3 T\,.$$

$$\sum_{\gamma\in S_{17}}\sum_{k\in A_3''}\frac{\Lambda(k)}{k^{1+\delta+i\gamma}}J_4(k)\ll\alpha T^{2/5}\log^3 T\,.$$

$$\sum_{\gamma\in S_{17}}\sum_{k\in A_3''}\frac{\Lambda(k)}{k^{1+\delta+i\gamma}}J_3(k)=(2\pi\alpha)^{b(1/2+\delta)}\sum_{1\ll k\ll\alpha^b}\frac{\Lambda(k)}{k^{1+\delta}}$$

$$\cdot\Bigg(\sum_{0<\gamma<\min(Y-T,\xi_k(1-\varepsilon))}e^{-i\gamma\log k}\Bigg)\cdot\xi_k^{1-b(1/2+\delta)}e^{-ib\xi_k}\int_0^{\varepsilon}e^{ib(\xi_k/2)\mu^2}d\mu$$

$$+O\Bigg(\alpha^{b/2}\sum_{1\ll k\ll\alpha^b}\frac{\Lambda(k)}{k^{1+\delta}}\Bigg|\sum_{0<\gamma<\min(Y-T,\xi_k(1-\varepsilon))}e^{-i\gamma\log k}\Bigg|\xi_k^{1-b(1/2+\delta)}(\varepsilon^2+\varepsilon^4\xi_k)\Bigg)$$

$$=M_1+M_2\,,\quad\text{say}\,.$$

$$M_1=(2\pi\alpha)^{b(1/2+\delta)}\sum_{1\ll k\ll\alpha^b}\frac{\Lambda(k)}{k^{1+\delta}}\xi_k^{1/2-b(1/2+\delta)}e^{-ib\xi_k}\frac{\sqrt{\pi}}{\sqrt{b}\,(1-i)}$$

$$\cdot\Bigg(\sum_{0<\gamma<\min(Y-T,\xi_k(1-\varepsilon))}e^{-i\gamma\log k}\Bigg)$$

$$+O\Bigg(\alpha^{b/2}\sum_{\substack{1\ll k\ll\alpha^b\\ \varepsilon\sqrt{\xi_k}\gg1}}\frac{\Lambda(k)}{k^{1+\delta}}\xi_k^{-b(1/2+\delta)}\frac{1}{\varepsilon}\Bigg|\sum_{0<\gamma<\min(Y-T,\xi_k(1-\varepsilon))}e^{-i\gamma\log k}\Bigg|\Bigg)$$

$$+O\Bigg(\alpha^{b/2}\sum_{\substack{1\ll k\ll\alpha^b\\ \varepsilon\sqrt{\xi_k}\ll1}}\frac{\Lambda(k)}{k^{1+\delta}}\cdot\xi_k^{1-b(1/2+\delta)}\varepsilon\Bigg|\sum_{0<\gamma<\min(Y-T,\xi_k(1-\varepsilon))}e^{-i\gamma\log k}\Bigg|\Bigg)$$

$$+O\Bigg(\alpha^{b/2}\sum_{\substack{1\ll k\ll\alpha^b\\ \varepsilon\sqrt{\xi_k}\ll1}}\frac{\Lambda(k)}{k^{1+\delta}}\xi_k^{1/2-b(1/2+\delta)}\Bigg|\sum_{0<\gamma<\min(Y-T,\xi_k(1-\varepsilon))}e^{-i\gamma\log k}\Bigg|\Bigg)$$

$$=M_3+M_4+M_5+M_6\,,\quad\text{say}\,.$$

$$M_5+M_6\ll\sqrt{\alpha}\sum_{\substack{1\ll k\ll\alpha^b\\ \varepsilon\sqrt{\xi_k}\ll1}}\frac{\Lambda(k)}{k^{(1+B)/2}}\Bigg(\frac{\alpha}{k^B}\frac{\Lambda(k)}{\sqrt{k}}+\sqrt{k}\log T\cdot\log\log T$$

$$+\sqrt{k}\log k\frac{\log T}{(\log\log T)^2}\Bigg)$$

$$\ll\sqrt{\alpha}\,T^{4/5}+T^{2/5}\alpha^b\log T\cdot\log\log T+T^{2/5}\alpha^b\log\alpha\frac{\log T}{(\log\log T)^2}\,.$$

$$M_4\ll T^{2/5-2b/5}\sum_{\substack{1\ll k\ll\alpha^b\\ \varepsilon\sqrt{\xi_k}\gg1}}\frac{\Lambda(k)}{k^{1+\delta}}\Bigg(\frac{\alpha}{k^B}\frac{\Lambda(k)}{\sqrt{k}}+\sqrt{k}\log T\cdot\log\log T+\sqrt{k}\log k\frac{\log T}{(\log\log T)^2}\Bigg)$$

$$\ll T^{2/5-2b/5}\Bigg(\alpha+\alpha^{b/2}\log T\cdot\log\log T+\alpha^{b/2}\log\alpha\frac{\log T}{(\log\log T)^2}\Bigg)\,.$$

$$\begin{aligned}M_3 &= -\sqrt{\frac{\alpha}{2b}}\frac{1}{1-i}\sum_{1\ll k\ll\alpha^b}\frac{\Lambda^2(k)}{k^{1+B/2}}e^{-ib\xi_k}\min(Y-T,\ \xi_k(1-\varepsilon))\\ &\quad+O\left(\sqrt{\alpha}\sum_{1\ll k\ll\alpha^b}\frac{\Lambda(k)}{k^{(B+1)/2}}\sqrt{k}\log T\log\log T\right)\\ &\quad+O\left(\sqrt{\alpha}\sum_{1\ll k\ll\alpha^b}\frac{\Lambda(k)}{k^{(B+1)/2}}\sqrt{k}\log k\frac{\log T}{(\log\log T)^2}\right)\\ &= -\sqrt{\frac{\alpha}{2b}}\frac{1}{1-i}\sum_{1\ll k\ll\alpha^b}\frac{\Lambda^2(k)}{k^{1+B/2}}e^{-ib\xi_k}\min(Y-T,\ \xi_k(1-\varepsilon))\\ &\quad+O(\Psi_1(\alpha, b)\log T\cdot\log\log T)+O\left(\Psi_2(\alpha, b)\frac{\log T}{(\log\log T)^2}\right)\\ &\ll\min((Y-T)\sqrt{\alpha},\ \alpha^{3/2})+\Psi_1(\alpha, b)\log T\cdot\log\log T+\Psi_2(\alpha, b)\frac{\log T}{(\log\log T)^2},\end{aligned}$$

where we put

$$\Psi_1(\alpha, b)=\begin{cases}\sqrt{\alpha} & \text{if } 0<b<\dfrac{1}{2}\\ \sqrt{\alpha}\log\alpha & \text{if } b=\dfrac{1}{2}\\ \alpha^b & \text{if } \dfrac{1}{2}<b\le 2\end{cases}$$

and

$$\Psi_2(\alpha, b)=\begin{cases}\sqrt{\alpha} & \text{if } 0<b<\dfrac{1}{2}\\ \sqrt{\alpha}\log^2\alpha & \text{if } b=\dfrac{1}{2}\\ \alpha^b\log\alpha & \text{if } \dfrac{1}{2}<b\le 2.\end{cases}$$

$$\begin{aligned}M_2 &\ll\alpha^{b/2}\sum_{1\ll k\ll\alpha^b}\frac{\Lambda(k)}{k^{1+\delta}}\left(\frac{\alpha}{k^B}\frac{\Lambda(k)}{\sqrt{k}}+\sqrt{k}\log T\cdot\log\log T+\sqrt{k}\log k\frac{\log T}{(\log\log T)^2}\right)\\ &\quad\cdot\left(\left(\frac{\alpha}{k^B}\right)^{1-b(1/2+\delta)}\left(T^{-4/5}+T^{-8/5}\frac{\alpha}{k^B}\right)\right)\\ &\ll\alpha^2T^{-4/5}+T^{-4/5}\log T\cdot\log\log T\cdot\Psi_3(\alpha, b)+T^{-4/5}\frac{\log T}{(\log\log T)^2}\Psi_4(\alpha, b)\\ &\quad+T^{-8/5}\alpha^3+T^{-8/5}\log T\cdot\log\log T\cdot\Psi_5(\alpha, b)+T^{-8/5}\frac{\log T}{(\log\log T)^2}\Psi_6(\alpha, b),\end{aligned}$$

were we put

$$\Psi_3(\alpha, b)=\begin{cases}\alpha & \text{if } 0<b<1\\ \alpha\log\alpha & \text{if } b=1\\ \alpha^b & \text{if } 1<b\le 2,\end{cases}$$

$$\Psi_4(\alpha, b)=\begin{cases}\alpha & \text{if } 0<b<1\\ \alpha\log^2\alpha & \text{if } b=1\\ \alpha^b\log\alpha & \text{if } 1<b\le 2,\end{cases}$$

$$\Psi_5(\alpha, b)=\begin{cases}\alpha^2 & \text{if } 0<b<1\\ \alpha^2\log\alpha & \text{if } b=1\\ \alpha^{3/2+b} & \text{if } 1<b\le 2\end{cases}$$

and

$$\Psi_6(\alpha, b)=\begin{cases}\alpha^2 & \text{if } 0<b<1\\ \alpha^2\log^2\alpha & \text{if } b=1\\ \alpha^{3/2+b}\log\alpha & \text{if } 1<b\le 2.\end{cases}$$

The sum $\sum_{\gamma\in S_{17}}\sum_{k\in A_3''}\frac{\Lambda(k)}{k^{1+\delta+i\gamma}}J_2(k)$ can be treated in the same manner and has the same expression as $\sum_{\gamma\in S_{17}}\sum_{k\in A_3''}\frac{\Lambda(k)}{k^{1+\delta+i\gamma}}J_3(k)$.

Hence we get

$$\begin{aligned}&\sum_{\gamma\in S_{17}}\sum_{k\in A_3''}\frac{\Lambda(k)}{k^{1+\delta+i\gamma}}J(k)\\ &\ll T^{2/5}\alpha\log^3 T+T^{2/5}\alpha^b\log T\log\log T+\sqrt{\alpha}\,T^{4/5}+T^{2/5}\alpha^b\log\alpha\frac{\log T}{(\log\log T)^2}\\ &\quad+T^{2/5-2b/5}\left(\alpha+\alpha^{b/2}\log T\cdot\log\log T+\alpha^{b/2}\log\alpha\frac{\log T}{(\log\log T)^2}\right)\\ &\quad+\min((Y-T)\sqrt{\alpha},\alpha^{3/2})+\Psi_1(\alpha, b)\log T\cdot\log\log T+\Psi_2(\alpha, b)\frac{\log T}{(\log\log T)^2}\\ &\quad+\alpha^2T^{-4/5}+T^{-4/5}\log T\cdot\log\log T\cdot\Psi_3(\alpha, b)+T^{-4/5}\frac{\log T}{(\log\log T)^2}\Psi_4(\alpha, b)\\ &\quad+T^{-8/5}\alpha^3+T^{-8/5}\log T\cdot\log\log T\cdot\Psi_5(\alpha, b)+T^{-8/5}\frac{\log T}{(\log\log T)^2}\Psi_6(\alpha, b).\end{aligned}$$

In a similar manner, we get the same upper bound to the sum

$$\sum_{\gamma\in S_{17}}\sum_{k\in(A_3'''\cup A_3')}\frac{\Lambda(k)}{k^{1+\delta+i\gamma}}J(k).$$

Treating the sum

$$\sum_{\gamma\in S_{17}}\sum_{k\in A_4}\frac{\Lambda(k)}{k^{1+\delta+i\gamma}}J(k)$$

in a similar manner, we get finally for $0<\alpha\ll T$,

$$\begin{aligned}S_9 \ll\, & T\log^3 T\cdot\log\log T+T^{2/5}(\alpha+3)\log^3 T+T^{2/5}(\alpha+3)^b\log T\cdot\log\log T\\ &+\sqrt{(\alpha+3)}\,T^{4/5}+T^{2/5}(\alpha+3)^b\log(\alpha+3)\frac{\log T}{(\log\log T)^2}\\ &+T^{2/5-2b/5}\left(\alpha+(\alpha+3)^{b/2}\log T\cdot\log\log T+(\alpha+3)^{b/2}\log(\alpha+3)\frac{\log T}{(\log\log T)^2}\right)\\ &+\min((Y-T)\sqrt{\alpha+3},\,(\alpha+3)^{3/2})+\Psi_1(\alpha,b)\log T\cdot\log\log T\\ &+\Psi_2(\alpha,b)\frac{\log T}{(\log\log T)^2}+(\alpha+3)^2T^{-4/5}+T^{-4/5}\log T\cdot\log\log T\cdot\Psi_3(\alpha,b)\\ &+T^{-4/5}\frac{\log T}{(\log\log T)^2}\Psi_4(\alpha,b)+T^{-8/5}(\alpha+3)^3+T^{-8/5}\log T\cdot\log\log T\cdot\Psi_5(\alpha,b)\\ &+T^{-8/5}\frac{\log T}{(\log\log T)^2}\Psi_6(\alpha,b)\,.\end{aligned}$$

§10. Proof of Theorems 6, 7 and 8—part (III)

In this section we shall evaluate S directly.

$$\begin{aligned}S=&-i\sum_{0<\gamma<Y-T}\sum_{k=2}^{\infty}\frac{\Lambda(k)}{k^{1+\delta-i\gamma}}\int_{\gamma}^{T+\gamma}\left(\frac{t}{2\pi\alpha}\right)^{b(1/2+\delta)}e^{ibt\log(t/2\pi e\alpha k^B)}dt\\ &+O\left(\alpha^{-b/2}\sum_{0<\gamma<Y-T}\sum_{k=2}^{\infty}\frac{\Lambda(k)}{k^{1+\delta}}\int_{\gamma+1}^{T+\gamma}t^{-1+b/2+\delta b}dt\right)\\ &+O(\alpha^{-b/2}T^{1+b/2}\log^2 T)\\ =&\,U_1+U_2+O(\alpha^{-b/2}T^{1+b/2}\log^2 T)\,,\quad\text{say}\,.\end{aligned}$$

We get trivially,

$$U_2\ll\alpha^{-b/2}T^{1+b/2}\log^2 T\,.$$

We shall evaluate U_1 in a different way from the previous sections.

$$U_1 = -ib^{-b(1/2+\delta)} \sum_{0<\gamma<Y-T} \sum_{2^j\gamma \le T+\gamma, j\ge 0} \sum_{k=2}^{\infty} \frac{\Lambda(k)}{k^{1+\delta-i\gamma}}$$

$$\cdot \int_{2^j\gamma}^{\min(2^{j+1}\gamma, T+\gamma)} \left(\frac{bt}{2\pi\alpha}\right)^{b(1/2+\delta)} e^{ibt\log(t/2\pi e\alpha k^B)} dt$$

$$= -ib^{-b(1/2+\delta)} \sum_{0<\gamma<Y-T} \sum_{2^j\gamma \le T+\gamma, j\ge 0} \sum_{2^j\gamma < 2\pi\alpha k^B \le \min(2^{j+1}\gamma, T+\gamma)} \frac{\Lambda(k)}{k^{1+\delta-i\gamma}}$$

$$\cdot e^{-2\pi i b\alpha k^B} \cdot \left(\frac{b}{2\pi}\right)^{b(1/2+\delta)-1/2} \frac{(2\pi\alpha k^B)^{b(1/2+\delta)+1/2}}{\alpha^{b(1/2+\delta)}} e^{\pi i/4}$$

$$+ O\left(b^{-(b(1/2+\delta))} \sum_{0<\gamma<Y-T} \sum_{2^j\gamma \le T+\gamma, j\ge 0} \sum_{k=2}^{\infty} \frac{\Lambda(k)}{k^{1+\delta-i\gamma}} E(2\pi\alpha k^B, 2^j\gamma, 2^{j+1}\gamma)\right)$$

$$= U_3 + U_4, \quad \text{say},$$

where we have used Lemma 2 of Fujii [19] and we put as in p. 130 of Gonek [21]

$$E(r, A, B) = O(A^{b(1/2+\delta)}) + O\left(\frac{A^{1+b(1/2+\delta)}}{|A-r| + \sqrt{A}}\right) + O\left(\frac{B^{1+b(1/2+\delta)}}{|B-r| + \sqrt{B}}\right).$$

It is easily seen that

$$U_3 = -2\pi i\sqrt{\frac{\alpha}{b}}\, e^{\pi i/4} \sum_{0<\gamma<Y-T} \sum_{(\gamma/2\pi\alpha)^b \le k \le ((T+\gamma)/2\pi\alpha)^b} \frac{\Lambda(k)k^{1/2b}}{\sqrt{k}\, k^{-i\gamma}} e^{-2\pi i b\alpha k^{1/b}}$$

$$= -2\pi i\sqrt{\frac{\alpha}{b}}\, e^{\pi i/4} \sum_{(C/2\pi\alpha)^b \le k \le (Y/2\pi\alpha)^b} \frac{\Lambda(k)k^{1/2b}}{\sqrt{k}} e^{-2\pi i b\alpha k^{1/b}} \sum_{\substack{2\pi\alpha k^B - T \le \gamma \le 2\pi\alpha k^B \\ 0<\gamma<Y-T}} k^{i\gamma}.$$

Applying Lemma 2 to the inner sum, we get

$$\sum_{\substack{2\pi\alpha k^B - T \le \gamma \le 2\pi\alpha k^B \\ 0<\gamma<Y-T}} k^{i\gamma}$$

$$= -\frac{1}{2\pi}(\min(2\pi\alpha k^B, Y-T) - \max(2\pi\alpha k^B - T, 0))\frac{\Lambda(k)}{\sqrt{k}}$$

$$+ O\left(\frac{\log T}{\log k} + \frac{\log T}{\log\log T} + \sqrt{k}\log k \log\log k + \sqrt{k}\,\log k \frac{\log T}{(\log\log T)^2}\right).$$

Hence, we get

$$U_3 = i\sqrt{\frac{\alpha}{b}}\, e^{\pi i/4} \sum_{(C/2\pi\alpha)^b \le k \le (Y/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}}$$

$$\cdot(\min(2\pi\alpha k^B,\, Y-T) - \max(2\pi\alpha k^B - T,\, 0))$$

$$+O\left(\sqrt{\alpha}\left(\frac{Y}{\alpha}\right)^{1/2+b/2}\left(\frac{\log T}{\log\log T}\right)^2\right)$$

$$= i\sqrt{\frac{\alpha}{b}}\, e^{\pi i/4} \sum_{(C/2\pi\alpha)^b \le k \le ((Y-T)/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}} \cdot 2\pi\alpha k^B$$

$$+ i\sqrt{\frac{\alpha}{b}}\, e^{\pi i/4} \sum_{((Y-T)/2\pi\alpha)^b < k \le (T/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}} \cdot (Y-T)$$

$$+ i\sqrt{\frac{\alpha}{b}}\, e^{\pi i/4} \sum_{(T/2\pi\alpha)^b < k \le (Y/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}} \cdot (Y-2\pi\alpha k^B)$$

$$+O\left(\sqrt{\alpha}\left(\frac{Y}{\alpha}\right)^{1/2+b/2}\left(\frac{\log T}{\log\log T}\right)^2\right).$$

To treat U_4, we use the following lemma which can be proved by modifying slightly the proof of Lemma 4 in Fujii [19].

LEMMA 8. *For $A < B \le 2A \ll T$ and for $\delta = \frac{1}{\log T}$, we have*

$$\sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{1+\delta}} E(2\pi\alpha n^{1/b}, A, B) \ll \left(\frac{A}{\alpha}\right)^{b/2} \log T \cdot \log\log T + \eta_\alpha(A)\left(\frac{\alpha}{A}\right)^{b/2} \sqrt{A}\, \log T,$$

where

$$\eta_\alpha(A) = \begin{cases} 1 & \text{if } \alpha < CA \\ 0 & \text{if } \alpha \ge CA \end{cases}$$

with some positive constant C.

Thus we get

$$U_4 \ll \sum_{0<\gamma<Y-T} \sum_{2^j\gamma \le T+\gamma,\, j\ge 0} \frac{(2^j\gamma)^{b/2}}{\alpha^{b/2}} \log T \cdot \log\log T$$

$$+ \sum_{0<\gamma<Y-T} \sum_{2^j\gamma \le T+\gamma,\, j\ge 0} \eta_\alpha(2^j\gamma)\left(\frac{\alpha}{2^j\gamma}\right)^{b/2} \sqrt{2^j\gamma}\, \log T$$

$$\ll \left(\frac{T}{\alpha}\right)^{b/2} T \log^3 T \cdot \log\log T$$

$$+ \sum_{0<\gamma<Y-T} \sum_{\alpha \ll 2^j\gamma \le T+\gamma,\, j \ge 0} \left(\frac{\alpha}{2^j\gamma}\right)^{b/2} \sqrt{2^j\gamma}\, \log T$$

$$\ll \left(\frac{T}{\alpha}\right)^{b/2} T\log^3 T \cdot \log\log T + \Psi_7(\alpha, b, T)\,,$$

where we put

$$\Psi_7(\alpha, b, T) = \begin{cases} \sqrt{\alpha}\, T\log^3 T & \text{if } 1 \le b \le 2 \\ \alpha^{b/2} T^{3/2-b/2} \log^3 T & \text{if } 0 < b < 1\,. \end{cases}$$

Consequently, we get

$$\begin{aligned} S = {} & i\sqrt{\frac{\alpha}{b}}\, e^{\pi i/4} \sum_{(C/2\pi\alpha)^b \le k \le ((Y-T)/2\pi\alpha)^b} \frac{\Lambda^2(k) k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}} \cdot 2\pi\alpha k^B \\ & + i\sqrt{\frac{\alpha}{b}}\, e^{\pi i/4} \sum_{((Y-T)/2\pi\alpha)^b < k \le (T/2\pi\alpha)^b} \frac{\Lambda^2(k) k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}} \cdot (Y-T) \\ & + i\sqrt{\frac{\alpha}{b}}\, e^{\pi i/4} \sum_{(T/2\pi\alpha)^b < k \le (Y/2\pi\alpha)^b} \frac{\Lambda^2(k) k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}} \cdot (Y - 2\pi\alpha k^B) \\ & + O\left(\left(\frac{T}{\alpha}\right)^{b/2} T\log^3 T \cdot \log\log T + \Psi_7(\alpha, b, T)\right). \end{aligned}$$

§11. Proof of Theorems 6, 7 and 8—part (IV)

In this section, we shall evaluate the following sum by the same method as above.

$$\tilde{S} \equiv \sum_{Y-T<\gamma\le T} \int_{1+\delta+i(\gamma+1)}^{1+\delta+iY} \frac{\zeta'}{\zeta}(s-i\gamma) e^{if(-i(s-1/2))} ds\,.$$

When $0 \le Y-T \le C$, we may take $Y = T + C$ below. Let $Y' = Y'(\gamma)$ be a number in $Y - \gamma \le Y' \le Y - \gamma + 1$ such that

$$\zeta(s) \ne 0 \qquad \text{in} \quad |\mathfrak{J}(s) - Y'| \le \frac{C}{\log T}\,.$$

We use the same notations with the tilders as in the sections 7, 8 and 9. Then we get

$$\begin{aligned} \tilde{S} = {} & \sum_{Y-T<\gamma\le T} \left(-\int_{1+\delta+iY'}^{-\delta+iY'} + \int_{-\delta+i(\gamma+1)}^{-\delta+iY'} - \int_{-\delta+i(\gamma+1)}^{1+\delta+i(\gamma+1)} \right) \cdot \frac{\zeta'}{\zeta}(s-i\gamma) e^{if(-i(s-1/2))} ds \\ & + 2\pi i \sum_{\substack{Y-T<\gamma\le T \\ \gamma+1<\gamma+\gamma'<Y}} e^{if(\gamma+\gamma')} + O(T\log^2 T) + O(T^{1+b/2}\log^2 T \cdot \alpha^{-b/2}) \\ = {} & \tilde{S}_1 + \tilde{S}_2 + \tilde{S}_3 + \tilde{S}_4 + O(T\log^2 T) + O(T^{1+b/2}\log^2 T \cdot \alpha^{-b/2})\,, \quad \text{say}\,. \end{aligned}$$

$$\tilde{S}_1 + \tilde{S}_3 \ll T^{1+b/2}\log^2 T \cdot \alpha^{-b/2} + T^{1-b/2}\log^3 T \cdot \alpha^{b/2}\,.$$

$$\tilde{S}_2 = i\log(2\pi)(2\pi\alpha)^{b(1/2+\delta)}\tilde{S}_{10} - i(2\pi\alpha)^{b(1/2+\delta)}\tilde{S}_{11} + \tilde{S}_9 + O(\alpha^{b/2}T^{1-b/2}\log^2 T)\,.$$

If $T^{4/5} \ll \alpha \ll T$, then we get

$$-i(2\pi\alpha)^{b(1/2+\delta)}\tilde{S}_{11} = -2i\frac{2\pi\alpha \cdot e^{-2\pi i\alpha b}}{\sqrt{2b\alpha}(1-i)} \sum_{Y-T<\gamma<\min(T,2\pi\alpha(1-\varepsilon))} \log\left(1-\frac{\gamma}{2\pi\alpha}\right)$$

$$-2i\frac{2\pi\alpha \cdot e^{-2\pi i\alpha b}\log(2\pi\alpha)}{\sqrt{2b\alpha}(1-i)} \sum_{Y-T<\gamma<\min(T,2\pi\alpha)} \cdot 1$$

$$+O(\alpha^{b/2}T^{1+2/5-b/2}\log T + T^{2/5}\alpha\log\alpha\cdot\log T + T^{-2/5}\alpha^2\log^2\alpha)$$

$$= O(\alpha^{3/2}\log^2\alpha + \alpha^{b/2}T^{1+2/5-b/2}\log T + T^{2/5}\alpha\log\alpha\cdot\log T + T^{-2/5}\alpha^2\log^2\alpha)\,.$$

If $0 < \alpha \ll T^{4/5}$, then we get

$$-i(2\pi\alpha)^{b(1/2+\delta)}\tilde{S}_{11} = O(\alpha^{b/2}T^{1+2/5-b/2}\log T + T^{2/5}(\alpha+3)\log(\alpha+3)\cdot\log T)\,.$$

In a similar manner, we can evaluate the sum $\tilde{S}_{10}$.

If $T^{4/5} \ll \alpha \ll T$, then we get

$$i\log(2\pi)(2\pi\alpha)^{b(1/2+\delta)}\tilde{S}_{10} = 2i\log(2\pi)\frac{2\pi\alpha\cdot e^{-2\pi i\alpha b}}{\sqrt{2b\alpha}(1-i)} \sum_{Y-T<\gamma<2\pi\alpha} \cdot 1$$

$$+O(\alpha^{b/2}T^{1+2/5-b/2}\log T + T^{2/5}\alpha\log\alpha\cdot\log T + T^{-2/5}\alpha^2\log^2\alpha)$$

$$= O(\alpha^{3/2}\log\alpha + \alpha^{b/2}T^{1+2/5-b/2}\log T + T^{2/5}\alpha\log\alpha\cdot\log T + T^{-2/5}\alpha^2\log^2\alpha)\,.$$

If $0 < \alpha \ll T^{4/5}$, then we get

$$i\log(2\pi)(2\pi\alpha)^{b(1/2+\delta)}\tilde{S}_{10} = O(\alpha^{b/2}T^{1+2/5-b/2}\log T + T^{2/5}(\alpha+3)\log(\alpha+3)\cdot\log T)\,.$$

As in the section 8, we get for $0 < \alpha \ll T$,

$$\tilde{S}_9 \ll T\log^3 T\cdot\log\log T + T^{2/5}(\alpha+3)\log^3 T + T^{2/5}(\alpha+3)^b\log T\cdot\log\log T$$

$$+\sqrt{(\alpha+3)}\,T^{4/5} + T^{2/5}(\alpha+3)^b\log(\alpha+3)\frac{\log T}{(\log\log T)^2}$$

$$+T^{2/5-2b/5}\left(\alpha+(\alpha+3)^{b/2}\log T\cdot\log\log T + (\alpha+3)^{b/2}\log(\alpha+3)\frac{\log T}{(\log\log T)^2}\right)$$

$$+\min((Y-T)\sqrt{\alpha+3},\,(\alpha+3)^{3/2}) + \Psi_1(\alpha,b)\log T\log\log T$$

$$+\Psi_2(\alpha,b)\frac{\log T}{(\log\log T)^2} + (\alpha+3)^2T^{-4/5} + T^{-4/5}\log T\cdot\log\log T\cdot\Psi_3(\alpha,b)$$

$$+T^{-4/5}\frac{\log T}{(\log\log T)^2}\Psi_4(\alpha,b) + T^{-8/5}(\alpha+3)^3 + T^{-8/5}\log T\cdot\log\log T\cdot\Psi_5(\alpha,b)$$

$$+T^{-8/5}\frac{\log T}{(\log\log T)^2}\Psi_6(\alpha,b)\,.$$

On the other hand, we shall evaluate $\tilde{S}$ directly as in the section 9.

$$\tilde{S} = -i \sum_{Y-T<\gamma\le T} \sum_{k=2}^{\infty} \frac{\Lambda(k)}{k^{1+\delta-i\gamma}} \int_{\gamma}^{Y} \left(\frac{t}{2\pi\alpha}\right)^{b(1/2+\delta)} e^{ibt\log(t/2\pi e\alpha k^B)}dt$$
$$+O(\alpha^{-b/2}T^{1+b/2}\log^2 T)$$
$$=\tilde{U}_1+O(\alpha^{-b/2}T^{1+b/2}\log^2 T), \quad \text{say}.$$

We get as in the section 9,

$$\tilde{U}_1 = -ib^{-b(1/2+\delta)} \sum_{Y-T<\gamma\le T} \sum_{2^j\gamma\le Y, j\ge 0} \sum_{2^j\gamma<2\pi\alpha k^B\le \min(2^{j+1}\gamma, Y)} \frac{\Lambda(k)}{k^{1+\delta-i\gamma}} e^{-2\pi i\alpha k^B}$$
$$\cdot \left(\frac{b}{2\pi}\right)^{b(1/2+\delta)-1/2} \frac{(2\pi\alpha k^B)^{b(1/2+\delta)+1/2}}{\alpha^{b(1/2+\delta)}} e^{\pi i/4}$$
$$+O\left(b^{-(b(1/2+\delta))} \sum_{Y-T<\gamma\le T} \sum_{2^j\gamma\le Y, j\ge 0} \sum_{k=2}^{\infty} \frac{\Lambda(k)}{k^{1+\delta-i\gamma}} E(2\pi\alpha k^B, 2^j\gamma, 2^{j+1}\gamma)\right)$$
$$=\tilde{U}_3+\tilde{U}_4, \quad \text{say}.$$

By the same analysis as in the previous section, we get

$$\tilde{U}_4 \ll \left(\frac{T}{\alpha}\right)^{b/2} T\log^3 T\cdot\log\log T+\Psi_7(\alpha, b, T).$$

$$\tilde{U}_3 = -2\pi i\sqrt{\frac{\alpha}{b}}\, e^{\pi i/4} \sum_{Y-T\le\gamma\le T} \sum_{(\gamma/2\pi\alpha)^b\le k\le (Y/2\pi\alpha)^b} \frac{\Lambda(k)k^{1/2b}}{\sqrt{k}\,k^{-i\gamma}} e^{-2\pi ib\alpha k^{1/b}}$$
$$= -2\pi i\sqrt{\frac{\alpha}{b}}\, e^{\pi i/4} \sum_{((Y-T)/2\pi\alpha)^b\le k\le (Y/2\pi\alpha)^b} \frac{\Lambda(k)k^{1/2b}}{\sqrt{k}} e^{-2\pi ib\alpha k^{1/b}} \sum_{\substack{0<\gamma\le 2\pi\alpha k^B\\ Y-T<\gamma\le T}} k^{i\gamma}.$$

Applying Lemma 2 to the inner sum, we get

$$\sum_{\substack{0<\gamma\le 2\pi\alpha k^B\\ Y-T<\gamma\le T}} k^{i\gamma}$$
$$=-\frac{1}{2\pi}(\min(2\pi\alpha k^B, T)-(Y-T))\frac{\Lambda(k)}{\sqrt{k}}$$
$$+O\left(\frac{\log T}{\log k}+\frac{\log T}{\log\log T}+\sqrt{k}\log k\cdot\log\log k+\sqrt{k}\log k\frac{\log T}{(\log\log T)^2}\right).$$

Hence, we get

$$\tilde{U}_3 = i\sqrt{\frac{\alpha}{b}}\,e^{\pi i/4} \sum_{((Y-T)/2\pi\alpha)^b \le k \le (T/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}} \cdot (\min(2\pi\alpha k^B, T) - (Y-T))$$

$$+ i\sqrt{\frac{\alpha}{b}}\,e^{\pi i/4} \sum_{(T/2\pi\alpha)^b \le k \le (Y/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}} \cdot (\min(2\pi\alpha k^B, T) - (Y-T))$$

$$+ O\left(\sqrt{\alpha}\left(\frac{T}{\alpha}\right)^{1/2+b/2}\left(\frac{\log T}{\log\log T}\right)^2\right)$$

$$= i\sqrt{\frac{\alpha}{b}}\,e^{\pi i/4} \sum_{((Y-T)/2\pi\alpha)^b \le k \le (T/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}} \cdot (2\pi\alpha k^B - Y + T)$$

$$+ i(2T - Y)\sqrt{\frac{\alpha}{b}}\,e^{\pi i/4} \sum_{(T/2\pi\alpha)^b \le k \le (Y/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}}$$

$$+ O\left(\sqrt{\alpha}\left(\frac{T}{\alpha}\right)^{1/2+b/2}\left(\frac{\log T}{\log\log T}\right)^2\right).$$

Consequently, we get

$$\tilde{S} = i\sqrt{\frac{\alpha}{b}}\,e^{\pi i/4} \sum_{((Y-T)/2\pi\alpha)^b \le k \le (T/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}} \cdot (2\pi\alpha k^B - Y + T)$$

$$+ i(2T - Y)\sqrt{\frac{\alpha}{b}}\,e^{\pi i/4} \sum_{(T/2\pi\alpha)^b \le k \le (Y/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}}$$

$$+ O\left(\left(\frac{T}{\alpha}\right)^{b/2} T\log^3 T \cdot \log\log T + \Psi_7(\alpha, b, T)\right).$$

§12. Completion of the proof of Theorem 7

Combining all of the evaluations in the sections 8, 9, 10 and 11, we get the following two theorems first.

THEOREM 9 (On R.H.). *If* $0 < b \le 2$, $T_0 < T \le Y \le 2T$ *and* $T^{4/5} \ll \alpha \ll T$, *then we have*

$$\sum_{\substack{0<\gamma,\gamma'<T\\ \gamma+\gamma'<Y}} e^{ib(\gamma+\gamma')\log((\gamma+\gamma')/2\pi e\alpha)}$$

$$= \sqrt{\frac{\alpha^3}{b}}\,e^{\pi i/4} \sum_{k \le (T/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{3/2b}}{k} e^{-2\pi i b\alpha k^{1/b}}$$

$$+ \frac{T}{\pi}\sqrt{\frac{\alpha}{b}}\,e^{\pi i/4} \sum_{(T/2\pi\alpha)^b < k \le (Y/2\pi\alpha)^b} \frac{\Lambda^2(k)k^{1/2b}}{k} e^{-2\pi i b\alpha k^{1/b}}$$

$$-\sqrt{\frac{\alpha^3}{b}}e^{\pi i/4}\sum_{(T/2\pi\alpha)^b<k\le(Y/2\pi\alpha)^b}\frac{\Lambda^2(k)k^{3/2b}}{k}e^{-2\pi ib\alpha k^{1/b}}$$
$$+O\left(\left(\frac{T}{\alpha}\right)^{b/2}T\log^3 T\cdot\log\log T+\Psi_7(\alpha,b,T)\right)+O\Big(T^{2/5}(\alpha+3)\log^3 T$$
$$+T^{2/5}(\alpha+3)^b\log T\cdot\log\log T+\sqrt{(\alpha+3)}\,T^{4/5}$$
$$+T^{2/5}(\alpha+3)^b\log(\alpha+3)\frac{\log T}{(\log\log T)^2}+(\alpha+3)^{3/2}\log^2(\alpha+3)\Big)$$
$$+\Psi_1(\alpha,b)\log T\cdot\log\log T+\Psi_2(\alpha,b)\frac{\log T}{(\log\log T)^2}$$
$$+T^{-4/5}\log T\cdot\log\log T\cdot\Psi_3(\alpha,b)+T^{-4/5}\frac{\log T}{(\log\log T)^2}\Psi_4(\alpha,b)\Big)$$
$$+T^{-8/5}\log T\cdot\log\log T\cdot\Psi_5(\alpha,b)+T^{-8/5}\frac{\log T}{(\log\log T)^2}\Psi_6(\alpha,b)$$
$$+O(\alpha^{b/2}T^{1+2/5-b/2}\log T+T^{-2/5}\alpha^2\log^2\alpha)\,.$$

THEOREM 10 (On R.H.). *If* $0<b\le 2$, $T_o<T\le Y\le 2T$ *and* $0<\alpha\ll T^{4/5}$, *then we have*

$$\sum_{\substack{0<\gamma,\gamma'<T\\ \gamma+\gamma'<Y}}e^{ib(\gamma+\gamma')\log((\gamma+\gamma')/2\pi e\alpha)}$$
$$=\sqrt{\frac{\alpha^3}{b}}e^{\pi i/4}\sum_{(C/2\pi\alpha)^b\le k\le(T/2\pi\alpha)^b}\frac{\Lambda^2(k)k^{3/2b}}{k}e^{-2\pi ib\alpha k^{1/b}}$$
$$+\frac{T}{\pi}\sqrt{\frac{\alpha}{b}}e^{\pi i/4}\sum_{(T/2\pi\alpha)^b<k\le(Y/2\pi\alpha)^b}\frac{\Lambda^2(k)k^{1/2b}}{k}e^{-2\pi ib\alpha k^{1/b}}$$
$$-\sqrt{\frac{\alpha^3}{b}}e^{\pi i/4}\sum_{(T/2\pi\alpha)^b<k\le(Y/2\pi\alpha)^b}\frac{\Lambda^2(k)k^{3/2b}}{k}e^{-2\pi ib\alpha k^{1/b}}$$
$$+O\left(\left(\frac{T}{\alpha}\right)^{b/2}T\log^3 T\cdot\log\log T+\Psi_7(\alpha,b,T)\right)+O\Big(T^{2/5}(\alpha+3)\log^3 T$$
$$+T^{2/5}(\alpha+3)^b\log T\cdot\log\log T+\sqrt{(\alpha+3)}\,T^{4/5}$$
$$+T^{2/5}(\alpha+3)^b\log(\alpha+3)\frac{\log T}{(\log\log T)^2}+\alpha^{b/2}T^{1+2/5-b/2}\log T$$
$$+T^{-8/5}\log T\cdot\log\log T\cdot\Psi_5(\alpha,b)+T^{-8/5}\frac{\log T}{(\log\log T)^2}\Psi_6(\alpha,b)\Big)\,.$$

If we ignore the dependence on α, then we get a simpler formula which has been described in the introduction as Theorem 7. This completes the proof of Theorem 7.

§13. Completion of the proof of Theorem 6

Suppose that $K\geq 2$. Taking $Y=T$ and $b=\frac{1}{K}$ in Theorem 7, we get for a fixed α,

$$\sum_{\substack{0<\gamma,\gamma'<T\\ \gamma+\gamma'<T}} e^{i((\gamma+\gamma')/K)\log((\gamma+\gamma')/2\pi e\alpha K)}$$
$$=\alpha^{3/2}K^2 e^{\pi i/4}\sum_{K\leq(T/2\pi\alpha K)^{1/K}} \Lambda^2(k)k^{3K/2-1}e^{-2\pi i\alpha k^K}+O(T^{3/2-1/2K}\log^3 T)$$
$$+O(T^{1+1/2K}\log^3 T\cdot\log\log T)\,.$$

Suppose first that $\alpha=\frac{a}{q}$, $(a,q)=1$, a, $q\geq 1$. Then

$$\alpha^{3/2}K^2 e^{\pi i/4}\sum_{k\leq(T/2\pi\alpha K)^{1/K}} \Lambda^2(k)k^{3K/2-1}e^{-2\pi i(a/q)k^K}$$
$$=\alpha^{3/2}K^2 e^{\pi i/4}\sum_{\substack{h=1\\ (q,h)=1}}^{q} e\left(-\frac{a}{q}h\right)\sum_{\substack{k\leq(T/2\pi\alpha K)^{1/K}\\ k^K\equiv h(\mathrm{mod}\ q)}} \Lambda^2(k)k^{3K/2-1}$$
$$+O(T^{3/2-1/K}\log T)$$
$$=\frac{1}{\varphi(q)}\alpha^{3/2}K^2 e^{\pi i/4}\sum_{\chi;q}\sum_{\substack{h=1\\ (q,h)=1}}^{q}\bar{\chi}(h)e\left(-\frac{a}{q}h\right)\sum_{k\leq(T/2\pi\alpha K)^{1/K}} \Lambda^2(k)k^{3K/2-1}\chi^K(k)$$
$$+O(T^{3/2-1/K}\log T)\,,$$

where χ runs over all Dirichlet character mod q.

We notice that when $\chi^K=\chi_0$,

$$\sum_{k\leq X} \Lambda^2(k)k^{3K/2-1}\chi^K(k)=X^{3K/2}\left(\log X-\frac{2}{3K}\right)\frac{2}{3K}+O(X^{3K/2}e^{-C\sqrt{\log X}})\,.$$

And that when $\chi^K\neq\chi_o$, then we have

$$\sum_{k\leq X} \Lambda^2(k)k^{3K/2-1}\chi^K(k)=O(X^{3K/2}e^{-C\sqrt{\log X}})\,.$$

We can see these easily from the proof of Lemma 9 below.

Hence, we get

$$\frac{1}{\varphi(q)}\alpha^{3/2}K^2e^{\pi i/4}\sum_{\chi;q}\sum_{\substack{h=1\\(q,h)=1}}^{q}\bar{\chi}(h)e\left(-\frac{a}{q}h\right)\sum_{k\le(T/2\pi\alpha K)^{1/K}}\Lambda^2(k)k^{3K/2-1}\chi^K(k)$$

$$=\frac{1}{\varphi(q)}e^{\pi i/4}\frac{2}{3K^{3/2}}\left(\frac{T}{2\pi}\right)^{3/2}\left(\log\frac{T}{2\pi\frac{a}{q}K}-\frac{2}{3}\right)\sum_{\substack{\chi;q\\ \chi^K=\chi_o}}\sum_{\substack{h=1\\(q,h)=1}}^{q}\bar{\chi}(h)e\left(-\frac{a}{q}h\right)$$

$$+O(T^{3/2}e^{-C\sqrt{\log T}})\,.$$

Thus we get

$$\sum_{\substack{0<\gamma,\gamma'<T\\ \gamma+\gamma'<T}}e^{i((\gamma+\gamma')/K)\log((\gamma+\gamma')/2\pi e\alpha K)}$$

$$=\frac{1}{\varphi(q)}e^{\pi i/4}\frac{2}{3K^{3/2}}\left(\frac{T}{2\pi}\right)^{3/2}\left(\log\frac{T}{2\pi\frac{a}{q}K}-\frac{2}{3}\right)\sum_{\substack{\chi;q\\ \chi^K=\chi_o}}\sum_{\substack{h=1\\(q,h)=1}}^{q}\bar{\chi}(h)e\left(-\frac{a}{q}h\right)$$

$$+O(T^{3/2}e^{-C\sqrt{\log T}})\,.$$

The last sum can be written as

$$\sum_{\substack{\chi;q\\ \chi^K=\chi_o}}\chi(a)\bar{\tau}(\chi)\,,$$

where $\tau(\chi)$ is the Gauss sum introduced in the statement of Theorem 6.

When α is irrational, we use Vinogradov's estimate on

$$\sum_{p\le X}e(\alpha p^K)\log p$$

and get our (i) of Theorem 6.

In a similar manner, we can prove (ii) of Theorem 6.

For a rational $\alpha=\frac{a}{q}$, $(a,q)=1$, $a,q\ge 1$, using Theorem 7 with $Y=2T$, we get

$$\sum_{0<\gamma,\gamma'<T}e^{i((\gamma+\gamma')/K)\log((\gamma+\gamma')/2\pi e(a/q)K)}$$

$$=\sqrt{\alpha^3}K^2e^{\pi i/4}\sum_{k\le(T/2\pi\alpha K)^{1/K}}\Lambda^2(k)k^{3K/2-1}e^{-2\pi i\alpha k^K}$$

$$+\frac{T}{\pi}\sqrt{\alpha}\,Ke^{\pi i/4}\sum_{(T/2\pi\alpha K)^{1/K}<k\le(2T/2\pi\alpha K)^{1/K}}\Lambda^2(k)k^{K/2-1}e^{-2\pi i\alpha k^K}$$

$$-\sqrt{\alpha^3}K^2e^{\pi i/4}\sum_{(T/2\pi\alpha K)^{1/K}<k\le(2T/2\pi\alpha K)^{1/K}}\Lambda^2(k)k^{3K/2-1}e^{-2\pi i\alpha k^K}$$

$$+O(T^{(3-(1/K))/2}\log^3T)+O(T^{1+1/2K}\log^3T\cdot\log\log T)$$

$$=\frac{1}{\varphi(q)}e^{\pi i/4}\frac{2}{3K^{3/2}}\left(\frac{T}{2\pi}\right)^{3/2}\left\{\log\frac{T}{2\pi\frac{a}{q}K}-\frac{2}{3}\right\}\sum_{\substack{\chi;q\\ \chi^K=\chi_o}}\chi(a)\bar{\tau}(\chi)$$

$$+\frac{1}{\varphi(q)}e^{\pi i/4}\frac{2}{3K^{3/2}}\left(\frac{T}{2\pi}\right)^{3/2}\left\{6(\sqrt{2}-1)\log\frac{T}{2\pi\frac{a}{q}K}\right.$$

$$\left.+6\sqrt{2}\log 2-12\sqrt{2}+12\right\}\cdot\sum_{\substack{\chi;q\\ \chi^K=\chi_o}}\chi(a)\bar{\tau}(\chi)$$

$$+\frac{1}{\varphi(q)}e^{\pi i/4}\frac{2}{3K^{3/2}}\left(\frac{T}{2\pi}\right)^{3/2}\left\{(1-2\sqrt{2})\log\frac{T}{2\pi\frac{a}{q}K}\right.$$

$$\left.-2\sqrt{2}\log 2+\frac{4\sqrt{2}}{3}-\frac{2}{3}\right\}\cdot\sum_{\substack{\chi;q\\ \chi^K=\chi_o}}\chi(a)\bar{\tau}(\chi)+O(T^{3/2}e^{-C\sqrt{\log T}})$$

$$=\frac{1}{\varphi(q)}e^{\pi i/4}\frac{2}{3K^{3/2}}\left(\frac{T}{2\pi}\right)^{3/2}\left\{4(\sqrt{2}-1)\log\frac{T}{2\pi\frac{a}{q}K}\right.$$

$$\left.+4\sqrt{2}\log 2-\frac{32}{3}\sqrt{2}+\frac{32}{3}\right\}\cdot\sum_{\substack{\chi;q\\ \chi^K=\chi_o}}\chi(a)\bar{\tau}(\chi)+O(T^{3/2}e^{-C\sqrt{\log T}}),$$

where we notice that

$$\sum_{(T/2\pi\alpha K)^{1/K}<k\le(2T/2\pi\alpha K)^{1/K}}\Lambda^2(k)k^{K/2-1}e^{-2\pi i\alpha k^K}$$

$$=\frac{1}{\varphi(q)}\sum_{\substack{\chi;q\\ \chi^K=\chi_o}}\chi(a)\bar{\tau}(\chi)\sqrt{\frac{T}{2\pi\alpha K}}\frac{2}{K^2}\left\{(\sqrt{2}-1)\log\frac{T}{2\pi\alpha K}+\sqrt{2}\log 2\right.$$

$$\left.+2-2\sqrt{2}\right\}+O(\sqrt{T}e^{-C\sqrt{\log T}})$$

and

$$\sum_{(T/2\pi\alpha K)^{1/K}<k\le(2T/2\pi\alpha K)^{1/K}}\Lambda^2(k)k^{3K/2-1}e^{-2\pi i\alpha k^K}$$

$$=\frac{1}{\varphi(q)}\sum_{\substack{\chi;q\\ \chi^K=\chi_o}}\chi(a)\bar{\tau}(\chi)\left(\frac{T}{2\pi\alpha K}\right)^{3/2}\frac{2}{3K^2}\left\{(2\sqrt{2}-1)\log\frac{T}{2\pi\alpha K}+2\sqrt{2}\log 2\right.$$

$$\left.+\frac{2}{3}-\frac{4\sqrt{2}}{3}\right\}+O(T^{3/2}e^{-C\sqrt{\log T}}).$$

Hence, we get our (ii) of Theorem 6 for a rational α. The case for an irrational α is the same as above.

§14. Completion of the proof of Theorem 8

We shall first notice the following lemma.

LEMMA 9. *Let A be any number ≥ 1 and let ψ be a Dirichlet character mod $q \geq 1$. Then the Generalized Riemann Hypothesis for $L(s, \psi)$ is equivalent to the following relation*

$$\sum_{n \leq X} \Lambda(n)^2 n^{A-1} \psi(n) = \delta(\psi) \frac{X^A}{A} \left(\log X - \frac{1}{A} \right) + O(X^{A-1/2+\varepsilon})$$

for any positive ε, where we put

$$\delta(\psi) = \begin{cases} 1 & \textit{if } \psi \textit{ is the principal character} \\ 0 & \textit{otherwise}. \end{cases}$$

We shall prove this, for completeness.

$$\sum_{n \leq X} \Lambda(n)^2 n^{A-1} \psi(n) = \sum_{n \leq X} \log n \cdot \Lambda(n) n^{A-1} \psi(n) + \sum_{n \leq X} (\Lambda(n) - \log n) \Lambda(n) n^{A-1} \psi(n)$$

$$= V_1 + V_2, \quad \text{say}.$$

$$V_2 = \sum_{p^r \leq X, r \geq 2} (\log p - r \log p) \log p \cdot p^{r(A-1)} \psi(p^r)$$

$$\ll X^{A-1} \sum_{p^r \leq X, r \geq 2} \log^2 p \cdot r$$

$$\ll X^{A-1} \left(\sum_{p \leq \sqrt{X}} \log^2 p + \sum_{p \leq X^{1/3}} \log^2 p \sum_{3 \leq r \leq \log X / \log p} \cdot r \right)$$

$$\ll X^{A-1/2} \log X.$$

Now suppose that G.R.H. holds for $L(s, \psi)$. Then

$$V_1 = \int_2^X \log y \cdot y^{A-1} d(\delta(\psi) y + R(y, \psi))$$

$$= \delta(\psi) \frac{X^A}{A} \left(\log X - \frac{1}{A} \right) + O(X^{A-1/2+\varepsilon}),$$

where we put

$$\sum_{n \leq X} \Lambda(n) \psi(n) = \delta(\psi) X + R(X, \psi).$$

Conversely, suppose that

$$\sum_{n\leq X} \Lambda(n)^2 n^{A-1}\psi(n) = \delta(\psi)\frac{X^A}{A}\left(\log X - \frac{1}{A}\right) + O(X^{A-1/2+\varepsilon})$$

for any positive ε. Then

$$\sum_{n\leq X} \log n \cdot \Lambda(n) n^{A-1}\psi(n) = \delta(\psi)\frac{X^A}{A}\left(\log X - \frac{1}{A}\right) + O(X^{A-1/2+\varepsilon}) .$$

Now

$$\begin{aligned}
\sum_{n\leq X} \Lambda(n)\psi(n) &= \sum_{n\leq X} \frac{\log n \cdot \Lambda(n) \cdot n^{A-1} \cdot \psi(n)}{\log n \cdot n^{A-1}} \\
&= \int_2^X \frac{1}{\log y \cdot y^{A-1}}\, d\left(\delta(\psi)\frac{y^A}{A}\left(\log y - \frac{1}{A}\right) + O(y^{A-1/2+\varepsilon})\right) \\
&= \delta(\psi)X + O(X^{1/2+\varepsilon}) .
\end{aligned}$$

This implies G.R.H. for $L(s, \psi)$.

Thus we have proved our Lemma 9.

We shall now prove our Theorem 8. Assume that the last relation in the statement of Theorem 8 holds. Then for any $(a, q)=1$, we have

$$\begin{aligned}
&\sum_{k\leq X} \Lambda^2(k) k^{3K/2-1} e^{-2\pi i(a/q)k^K} \\
&= \frac{1}{\varphi(q)} X^{3K/2}\left(\log X - \frac{2}{3K}\right)\frac{2}{3K} \sum_{\substack{\chi;q \\ \chi^K=\chi_o}} \sum_{\substack{h=1 \\ (q,h)=1}}^{q} \bar{\chi}(h) e\left(-\frac{a}{q}h\right) + O(X^{3K/2-1/2+\varepsilon}) .
\end{aligned}$$

Thus for any Dirichlet character ψ mod q, we get

$$\begin{aligned}
&\bar{\tau}(\psi) \sum_{k\leq X} \Lambda^2(k) k^{3K/2-1} \psi^K(k) \\
&= \sum_{a=1}^{q} \sum_{k\leq X} \Lambda^2(k) k^{3K/2-1} e^{-2\pi i(a/q)k^K} \bar{\psi}(a) \\
&= \frac{1}{\varphi(q)} X^{3K/2}\left(\log X - \frac{2}{3K}\right)\frac{2}{3K} \sum_{a=1}^{q} \sum_{\substack{\chi;q \\ \chi^K=\chi_o}} \bar{\tau}(\chi)\chi(a)\bar{\psi}(a) + O(X^{3K/2-1/2+\varepsilon}) \\
&= X^{3K/2}\left(\log X - \frac{2}{3K}\right)\frac{2}{3K} \sum_{\substack{\chi;q \\ \psi=\chi, \chi^K=\chi_o}} \bar{\tau}(\chi) + O(X^{3K/2-1/2+\varepsilon}) .
\end{aligned}$$

By Lemma 9, this implies G.R.H. for $L(s, \psi^K)$ for all ψ mod q.

The converse is clear from the argument of the last section.

This completes the proof of Theorem 8.

We may notice here that Lemma 9 can be extended further. For example, the same method proves the following which does not seem to have appeared in the literature.

LEMMA 9′. *Let K be an integer ≥ 1. Then the Riemann Hypothesis is equivalent to the relation*

$$\sum_{n \leq X} \Lambda^K(n) = XP_K(\log X) + O(X^{1/2+\varepsilon})$$

for any positive ε, where we put

$$P_K(t) = t^{K-1} - (K-1)t^{K-2} + (K-1)(K-2)t^{K-3} - \cdots + (-1)^{K+1}(K-1)!\,.$$

§15. Concluding remarks

15-1. We can extend our results to

$$\gamma_1 + \gamma_2 + \cdots + \gamma_K$$

for $K \geq 3$. However, it becomes more complicated.

We shall give only a short notice for the Riemann-von Mangoldt formula for $\gamma_1 + \gamma_2 + \gamma_3$.

$$\begin{aligned} N_3(T) &\equiv \sum_{\substack{0<\gamma_1,\gamma_2,\gamma_3 \leq T \\ 0<\gamma_1+\gamma_2+\gamma_3 \leq T}} \cdot 1 = \sum_{0<\gamma_1 \leq T} \sum_{\substack{0<\gamma_2,\gamma_3 \leq T \\ 0<\gamma_2+\gamma_3 \leq T-\gamma_1}} \cdot 1 \\ &= \sum_{0<\gamma_1 \leq T} L_2(T-\gamma_1) + \sum_{0<\gamma_1 \leq T} R_2(T-\gamma_1) \\ &= L_3(T) + R_3(T)\,, \quad \text{say}\,. \end{aligned}$$

It is easily seen that

$$\begin{aligned} L_3(T) &\sim \int_1^{T-C} \frac{1}{8\pi^2} t^2 \log^2 t \cdot \frac{1}{2\pi} \log \frac{T-t}{2\pi}\, dt \\ &\sim \frac{1}{48\pi^3} T^3 \log^3 T\,. \end{aligned}$$

Under R.H., a trivial estimate, using $R_2(T) \ll T \log T$, gives

$$R_3(T) \ll T^2 \log^2 T\,.$$

On the other hand, we have also under R.H.,

$$\begin{aligned} \int_0^T R_3(t)dt &= \int_0^T \sum_{0<\gamma \leq t} R_2(t-\gamma)dt = \sum_{0<\gamma \leq T} \int_\gamma^T R_2(t-\gamma)dt \\ &= \sum_{0<\gamma \leq T} \int_0^{T-\gamma} R_2(t)dt \end{aligned}$$

$$= -\frac{1}{2\pi}\left(\frac{1}{\pi}\int_{1/2}^{\infty}\log|\zeta(\sigma)|d\sigma\right)\cdot\sum_{0<\gamma\le T}(T-\gamma)\log(T-\gamma)$$

$$-\frac{1}{2\pi^2}\Re\left\{\sum_{0<\gamma\le T}\log\zeta(1+i(T-\gamma))\cdot(T-\gamma)\right\}+O(T^2\log T).$$

It is easily seen that

$$-\frac{1}{2\pi}\left(\frac{1}{\pi}\int_{1/2}^{\infty}\log|\zeta(\sigma)|d\sigma\right)\cdot\sum_{0<\gamma\le T}(T-\gamma)\log(T-\gamma)$$

$$= -\left(\frac{1}{\pi}\int_{1/2}^{\infty}\log|\zeta(\sigma)|d\sigma\right)\cdot\frac{1}{8\pi^2}T^2\log^2 T+O(T^2\log T).$$

On the other hand, with $\delta=\frac{1}{\log T}$, we get by applying Lemma 2,

$$\sum_{0<\gamma\le T}\log\zeta(1+i(T-\gamma))\cdot(T-\gamma)$$

$$= \sum_{0<\gamma\le T}\log\zeta(1+i(T-\gamma))\int_{\gamma}^{T}dt=\int_0^T\sum_{0<\gamma\le t}\log\zeta(1+i(T-\gamma))dt$$

$$= \int_0^T\sum_{0<\gamma\le t}\log\zeta(1+\delta+i(T-\gamma))dt+O\left(T^2\log T\left(\frac{\log\log T}{\log T}\right)^{1/3}\right)$$

$$= \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{1+\delta}\log n\cdot n^{iT}}\int_0^T\sum_{0<\gamma\le t}n^{i\gamma}dt$$

$$= \sum_{n<T}\frac{\Lambda(n)}{n^{1+\delta}\log n\cdot n^{iT}}\int_0^T\left(-\frac{t}{2\pi}\frac{\Lambda(n)}{\sqrt{n}}+O\left(\frac{\log t}{\log n}\right)+O\left(\frac{\log T}{\log\log T}\right)\right.$$

$$\left.+O\left(\sqrt{n}\log n\frac{\log T}{(\log\log T)^2}\right)\right)dt+\sum_{n\ge T}\frac{\Lambda(n)}{n^{1+\delta}\log n\cdot n^{iT}}O(T^2\log T)$$

$$= O(T^2)+O\left(T^2\log T\int_T^{\infty}\frac{1}{y^{1+\delta}\log y}d(y+O(\sqrt{y}\log^2 y))\right)$$

$$= O(T^2\log T).$$

Thus we get under R.H.,

$$\int_0^T R_3(t)dt=-\left(\frac{1}{\pi}\int_{1/2}^{\infty}\log|\zeta(\sigma)|d\sigma\right)\cdot\frac{1}{8\pi^2}T^2\log^2 T+O(T^2\log T).$$

This should be compared with our Theorem 3 under R.H.

15-2. The estimate of S_9 in the section 9 and $\tilde{S}_9$ in the section 11 may be refined if one uses the method in the section 10. This will refine the dependence on α in Theorems 9 and 10 in the section 12.

However, if we fix α, then we get the same conclusion as is stated in Theorem 7 in the introduction.

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